4. Image Enhancement in the Frequency Domain

- Chapter 4 (Image Enhancement in the Frequency Domain)

Fourier Transform

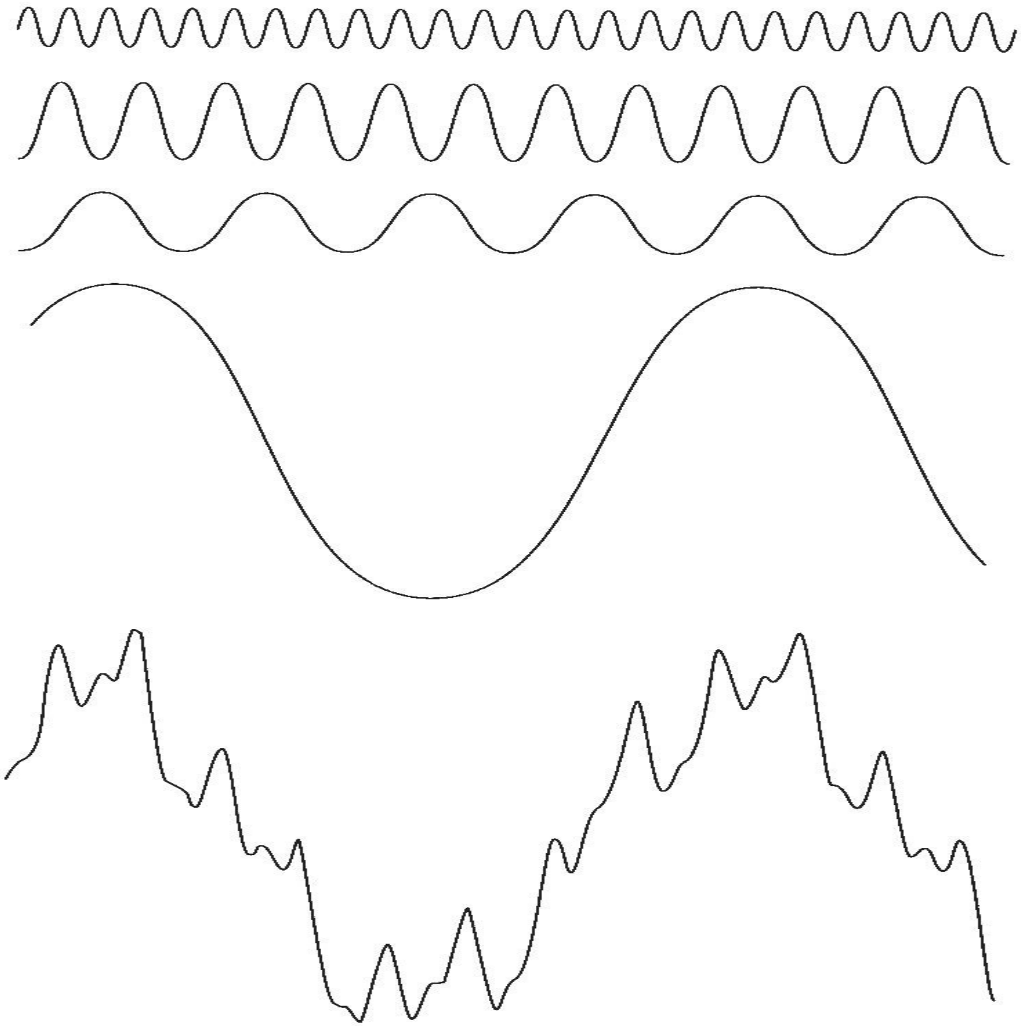


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Fourier Transform

- Fourier Transform of a real function is generally complex

$$F(f(x)) = F(u) = \int_{-\infty}^{\infty} f(x)\exp[\ j2\pi ux]dx = R(u) + jI(u) = \left|F(u)\right|e^{j\Phi(u)}$$

$$F^{-1}(F(u)) = f(x) = \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du$$

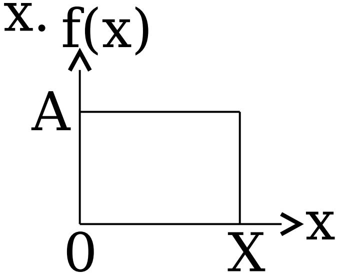
$$\exp[\ j2\pi ux] = \cos 2\pi ux - j\sin 2\pi ux, \quad j = \sqrt{-1}$$

$$\left|F(u)\right| = \left[R^2(u) + I^2(u)\right]^{\frac{1}{2}}$$ Fourier Spectru

$$\Phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$ phas

$$\left|F(u)\right|^2 = F(u)F^*(u) = R^2(u) + I^2(u)$$ Power Spectru

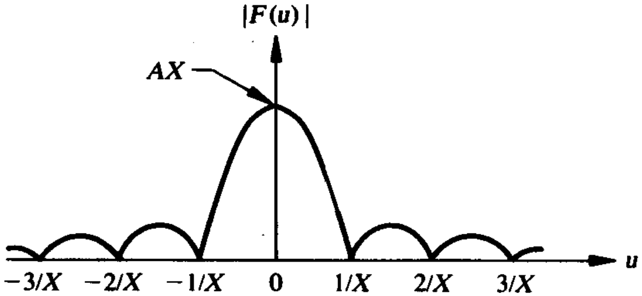
Ex.



$$F(u) = \int_0^X A\exp[-j2\pi ux]dx = \frac{-A}{j2\pi u}[\exp[-j2\pi ux]]_0^X$$

$$= \frac{A}{\pi u}\sin(\pi uX)\exp[-j\pi uX]$$

$$\left|F(u)\right| = AX\left|\frac{\sin(\pi uX)}{\pi uX}\right|$$



Some Properties of Fourier Transform

Convolution:

$$f(x)\star g(x)=\int_{-\infty}^{\infty} f(\alpha)g(x-\alpha)\,d\alpha$$

$$f(x)\star g(x)\leftrightarrow F(u)G(u)$$

$$f(x)g(x)\leftrightarrow F(u)\star G(u)$$

The convolution in spatial domain corresponds to the multiplication in frequency domain.

Correlation:

$$f(x)\circ g(x)=\int_{-\infty}^{\infty} f^{*}(\alpha)g(x+\alpha)\,d\alpha$$

$$f(x)\circ g(x)\leftrightarrow F^{*}(u)G(u)$$

$$f^{*}(x)g(x)\leftrightarrow F(u)\circ G(u)$$

$$f(x)\circ f(x)\leftrightarrow F^{*}(u)F(u)=\left|F(u)\right|^{2}$$ power spectrum

2-D Fourier Transform

$$F(f(x,y)) = F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-j2\pi(ux+vy)]dxdy = \left|F(u,v)\right| e^{j\Phi(u,v)}$$

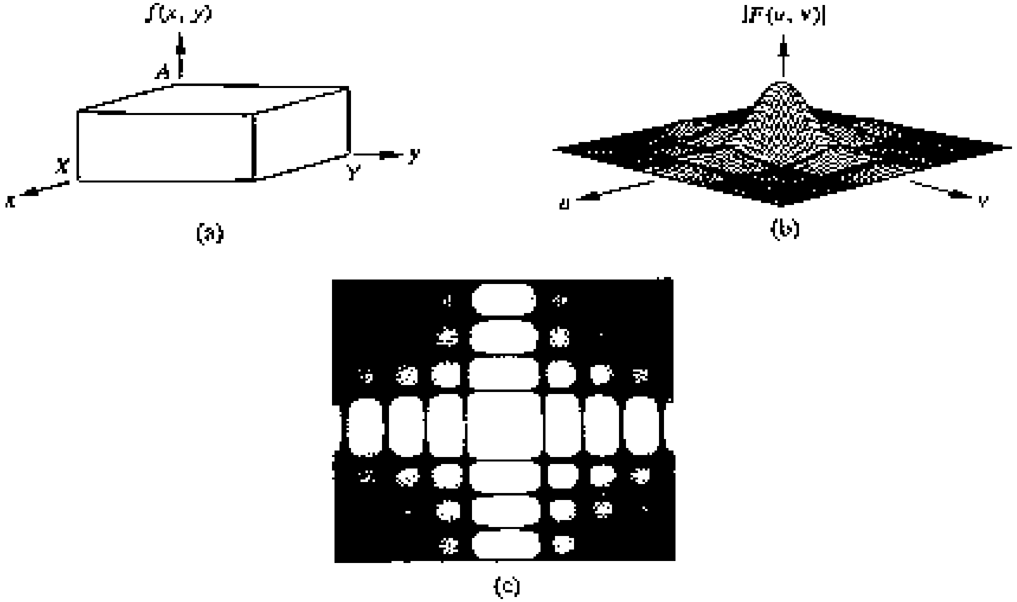
$$F^{-1}(F(u,v)) = f(x,y) = \int_{-\infty}^{\infty} F(u,v)\exp[j2\pi(ux+vy)]dudv$$

$$\left|F(u,v)\right| = \left[R^2(u,v) + I^2(u,v)\right]^{\frac{1}{2}}$$

$$\Phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right],$$

$$\left|F(u,v)\right|^2 = F(u,v)F^*(u,v)$$

$$= R^2(u,v) + I^2(u,v)$$



(a)

(b)

(c)

Discrete Fourier Transform (DFT)

- 1-D DFT pair for a signal sequence x(n) of length N:

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}, \quad k = 0,\ldots,N-1$$

$$x(n) = \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi nk}{N}}, \quad n = 0,\ldots,N-1.$$

2-D DFT pair for a signal sequence x(n$_1$,n$_2$) of area NxN:

$$X(k_1,k_2) = \frac{1}{N^2}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} x(n_1,n_2)\, e^{\frac{-j2\pi(n_1k_1+n_2k_2)}{N}}, \quad k_1,k_2 = 0,\ldots,N-1,$$

$$x(n_1,n_2) = \sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} X(k_1,k_2)\, e^{\frac{j2\pi(n_1k_1+n_2k_2)}{N}}, \quad n_1,n_2 = 0,\ldots,N-1$$

ometimes $\frac{1}{N^2}$ appears in inverse, or $\frac{1}{N}$ appears in each s

$$X(0,0) = \frac{1}{N^2}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} x(n_1,n_2) = \bar{x}(n_1,n_2)$$ DC component

Example: 1-D DFT

$$X(k)=\frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{\frac{-j2\pi nk}{N}},\ k=0,\ldots,N-1$$

If f(n) = { 2, 3, 4, 4}:

$$F(0)=\frac{1}{4}\sum_{n=0}^{3}f(n)e^{0}=\frac{1}{4}[2+3+4+4]=3.25$$

$$F(1)=\frac{1}{4}\sum_{n=0}^{3}f(n)e^{-j2\pi n/4}$$

$$=\frac{1}{4}[2+3e^{-j\pi/2}+4e^{-j\pi}+4e^{-j3\pi/2}]=-0.5+j0.25$$

$$F(2)=-0.25$$

$$F(3)=-0.5-j0.25$$

Example: 1-D IDFT

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi nk}{N}}, \; n = 0, \ldots, N-1.$$

$$f(0) = \sum_{k=0}^{3} F(k) e^{0} = 3.25 + (-0.5 + j0.25) + (-0.25) + (-0.5 - j0.25)$$

$$= 2$$

$$f(1) = \sum_{k=0}^{3} F(k) e^{j2\pi k/4}$$

$$= 3.25 + (-0.5 + j0.25) e^{j\pi/2} + (-0.25) e^{j\pi} + (-0.5 - j0.25) e^{j3\pi/2}$$

$$= 3.25 + (-0.5 + j0.25)(0 + j) + (-0.25)(-1 + j0) + (-0.5 - j0.25)(0 - j)$$

$$= 3$$

$$f(2) = \sum_{k=0}^{3} F(k) e^{j2\pi 2k/4}$$

$$= 3.25 + (-0.5 + j0.25) e^{j\pi} + (-0.25) e^{j2\pi} + (-0.5 - j0.25) e^{j3\pi}$$

$$= 3.25 + (-0.5 + j0.25)(-1 + j0) + (-0.25)(1 + j0) + (-0.5 - j0.25)(-1 + j0)$$

$$= 4$$

$$f(3) = \sum_{k=0}^{3} F(k) e^{j2\pi 3k/4}$$

$$= 3.25 + (-0.5 + j0.25) e^{j3\pi/2} + (-0.25) e^{j3\pi} + (-0.5 - j0.25) e^{j\pi/2}$$

$$= 3.25 + (-0.5 + j0.25)(0 - j) + (-0.25)(-1 + j0) + (-0.5 - j0.25)(0 + j)$$

$$= 4$$

Example: 1-D IDFT

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi nk}{N}}, \;\; n = 0, \ldots, N-1.$$

$$f(n) = (2, 3, 4, 4)$$

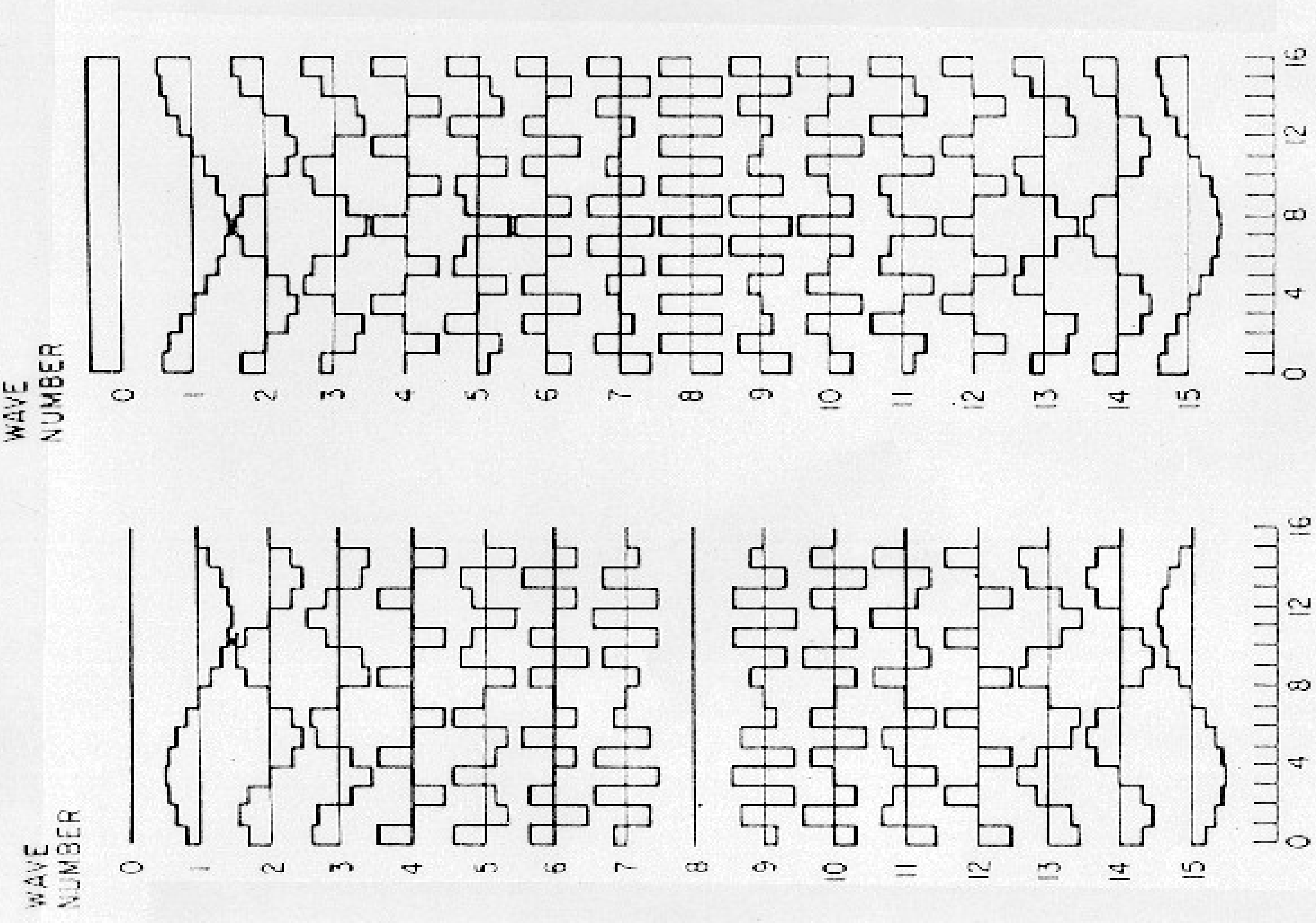
$$= F(0)(1 + j0,\; 1 + j0,\; 1 + j0,\; 1 + j0)$$

$$+ F(1)(1 + j0,\; 0 + j1,\; -1 + j0,\; 0 - j1)$$

$$+ F(2)(1 + j0, -1 + j0,\; 1 + j0,\; -1 + j0)$$

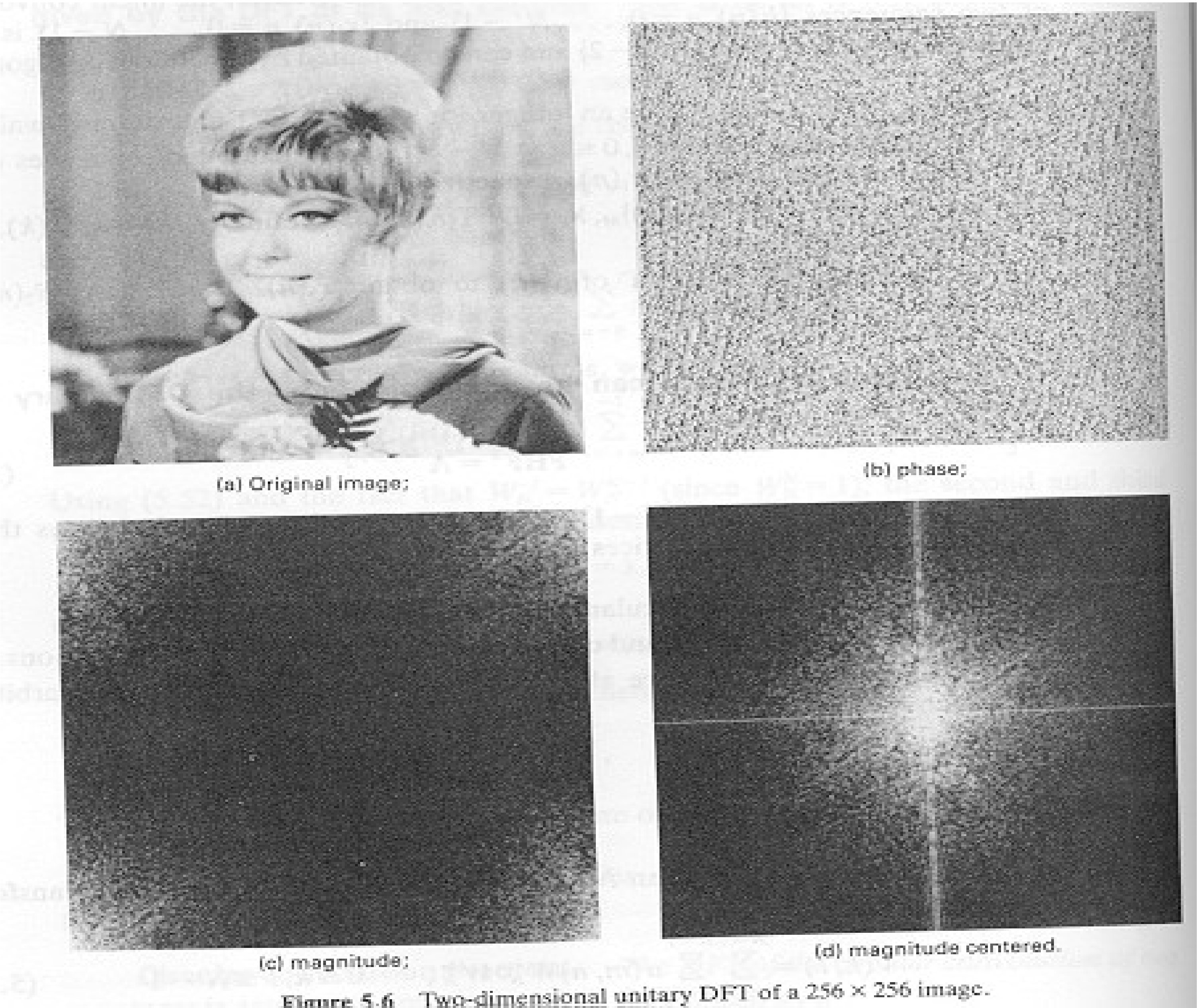
$$+ F(3)(1 + j0,\; 0 - j1,\; -1 + j0,\; 0 + j1)$$

1-D 16-POINT DFT BASIS VECTORS



Fourier basis functions. (a) Sine component. (b) Cosine component.

2-D DFT of A Typical Image



(a) Original image;

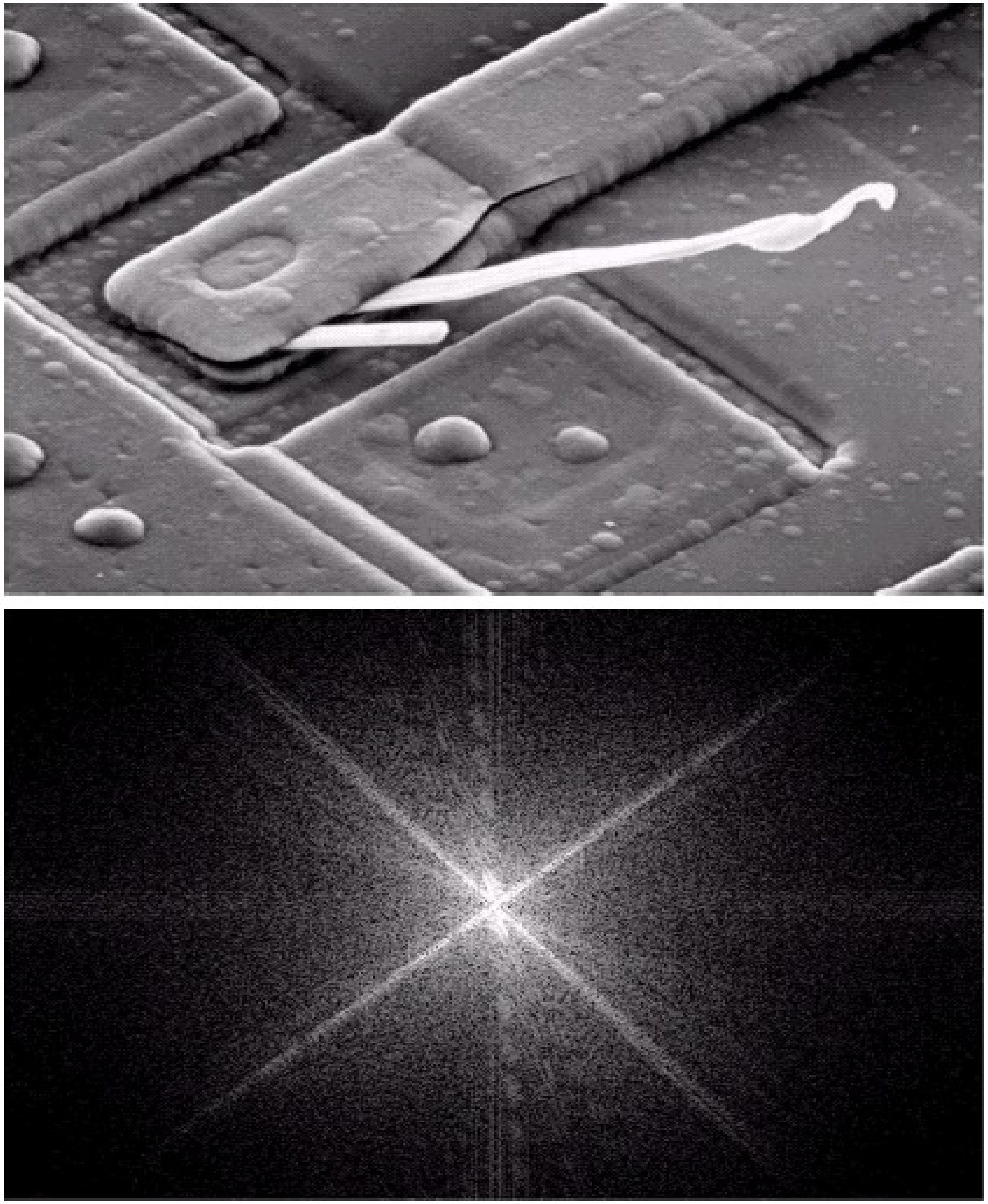
(b) phase;

(c) magnitude;

(d) magnitude centered.

Figure 5.6 Two-dimensional unitary DFT of a 256 × 256 image.

Another 2-D DFT Example



a
b

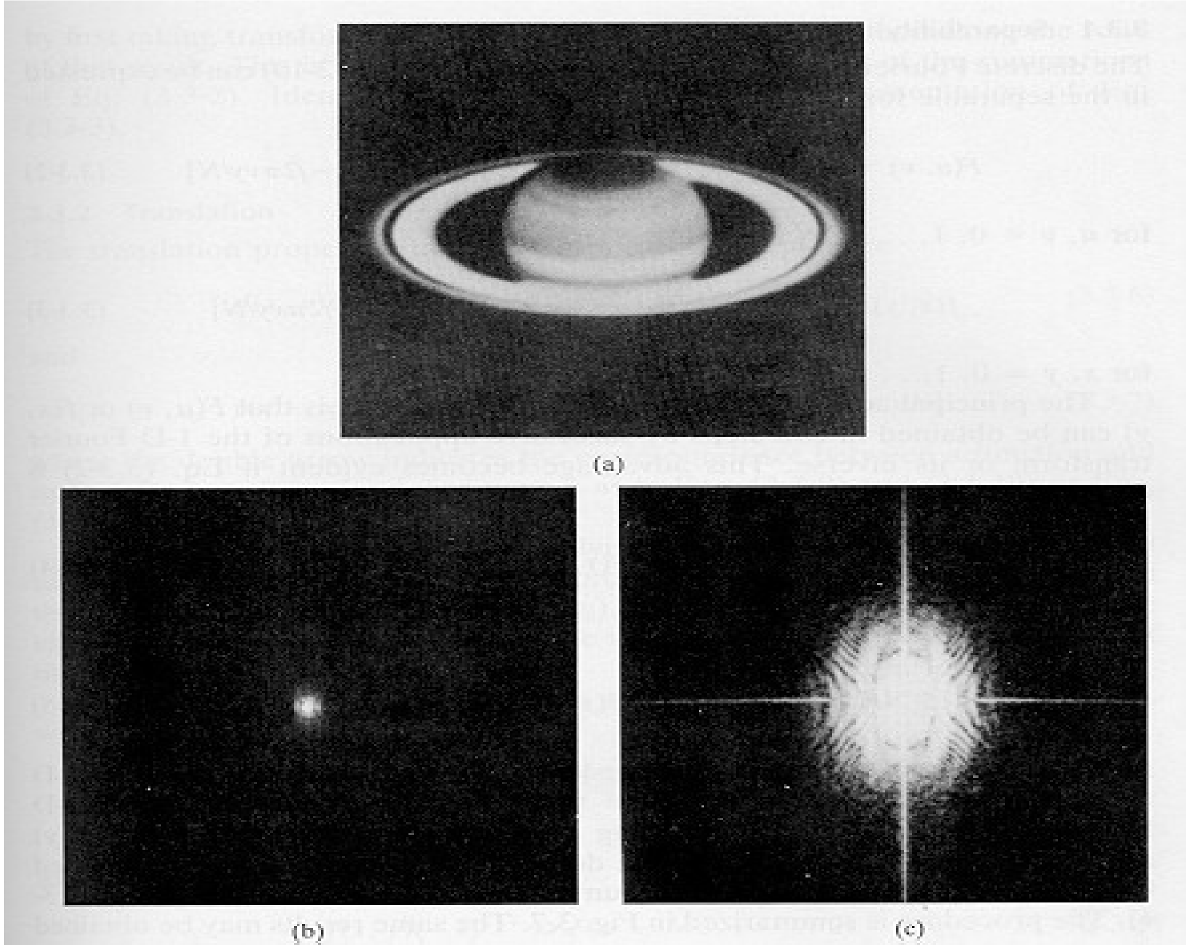
FIGURE 4.4
(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Fourier Spectra

dynamic range of Fourier Spectra is usually much higher t

cal display devices to display. For display, we often use:

$$D(u,v) = C \bullet \log(1 + |F(u,v)|),$$ where C is a scaling constant.



(a)

(b)

(c)

Useful Properties of 2-D DFT

- Periodic extension in “**space**” and “frequency”.
- **Linearity:**

$$a x_1(n_1,n_2) + b x_2(n_1,n_2) \Rightarrow aX_1(k_1,k_2) + bX_2(k_1,k_2)$$

- **Circular Convolution:**

$$x_1(n_1,n_2) \star x_2(n_1,n_2) \Rightarrow X_1(k_1,k_2) \bullet X_2(k_1,k_2)$$

- **Multiplication:**

$$x_1(n_1,n_2) \bullet x_2(n_1,n_2) \Rightarrow X_1(k_1,k_2) \star X_2(k_1,k_2)$$

- **Shift (Translation):**

$$x(n_1 - m_1, n_2 - m_2) \Rightarrow X(k_1,k_2)\, e^{\frac{-j2\pi(m_1k_1+m_2k_2)}{N}}$$

- **Periodic in Frequency Domain**

$$X(k_1,k_2) = X(k_1+N,k_2) = X(k_1,k_2+N) = X(k_1+N,k_2+N)$$

- **Conjugate Symmetry:**

$$X(k_1,k_2) = X^*(-k_1,-k_2) = X^*(N-k_1,N-k_2)$$

2-D Convolution and Correlation

- 2-D Convolution

$$f(x,y)*h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$$

$$f(x,y)\star h(x,y)\Leftrightarrow F(u,v)H(u,v)$$

$$f(x,y)h(x,y)\Leftrightarrow F(u,v)\star H(u,v)$$

- 2-D Correlation

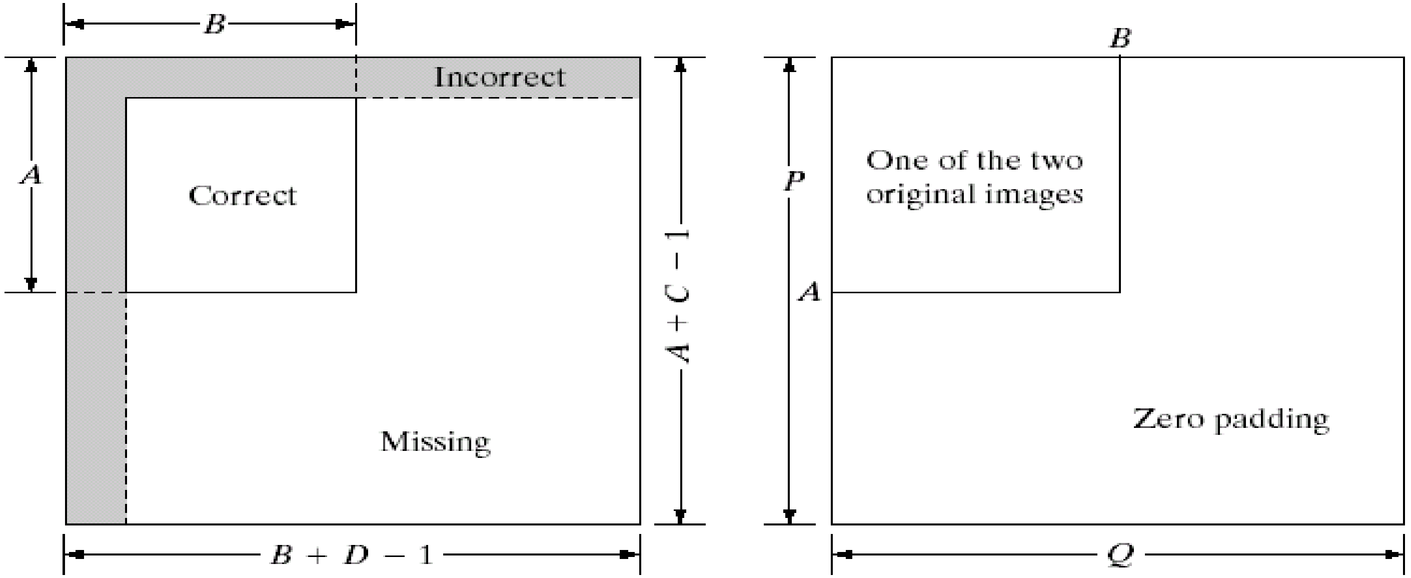
$$f(x,y)\circ h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^{*}(m,n)h(x+m,y+n)$$

$$f(x,y)\circ h(x,y)\Leftrightarrow F^{*}(u,v)H(u,v)$$

$$f^{*}(x,y)g(x,y)\Leftrightarrow F(u,v)\circ G(u,v)$$

$$f(x,y)\circ f(x,y)\Leftrightarrow F^{*}(u,v)F(u,v)=\left|F(u,v)\right|^{2},\ \text{power spectrum}$$

Zero Padding Before DFT (convolution)

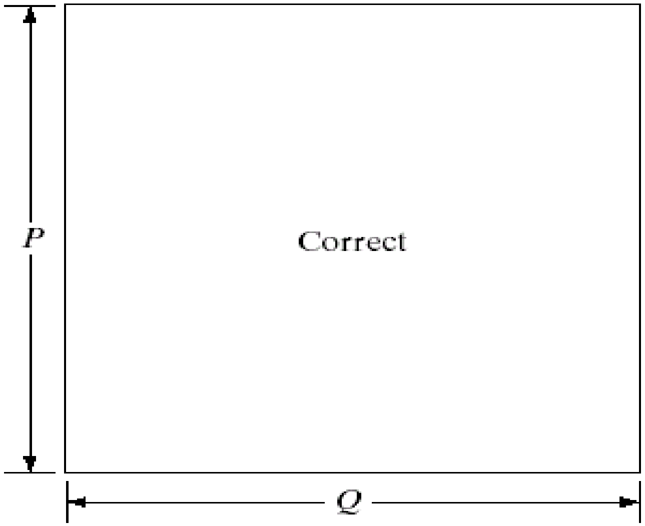


Result of filtering in the frequency domain without properly padding the input images.

Properly extended (padded) image

a b
c

FIGURE 4.38 Illustration of the need for function padding. (a) Result of performing 2-D convolution without padding. (b) Proper function padding. (c) Correct convolution result.



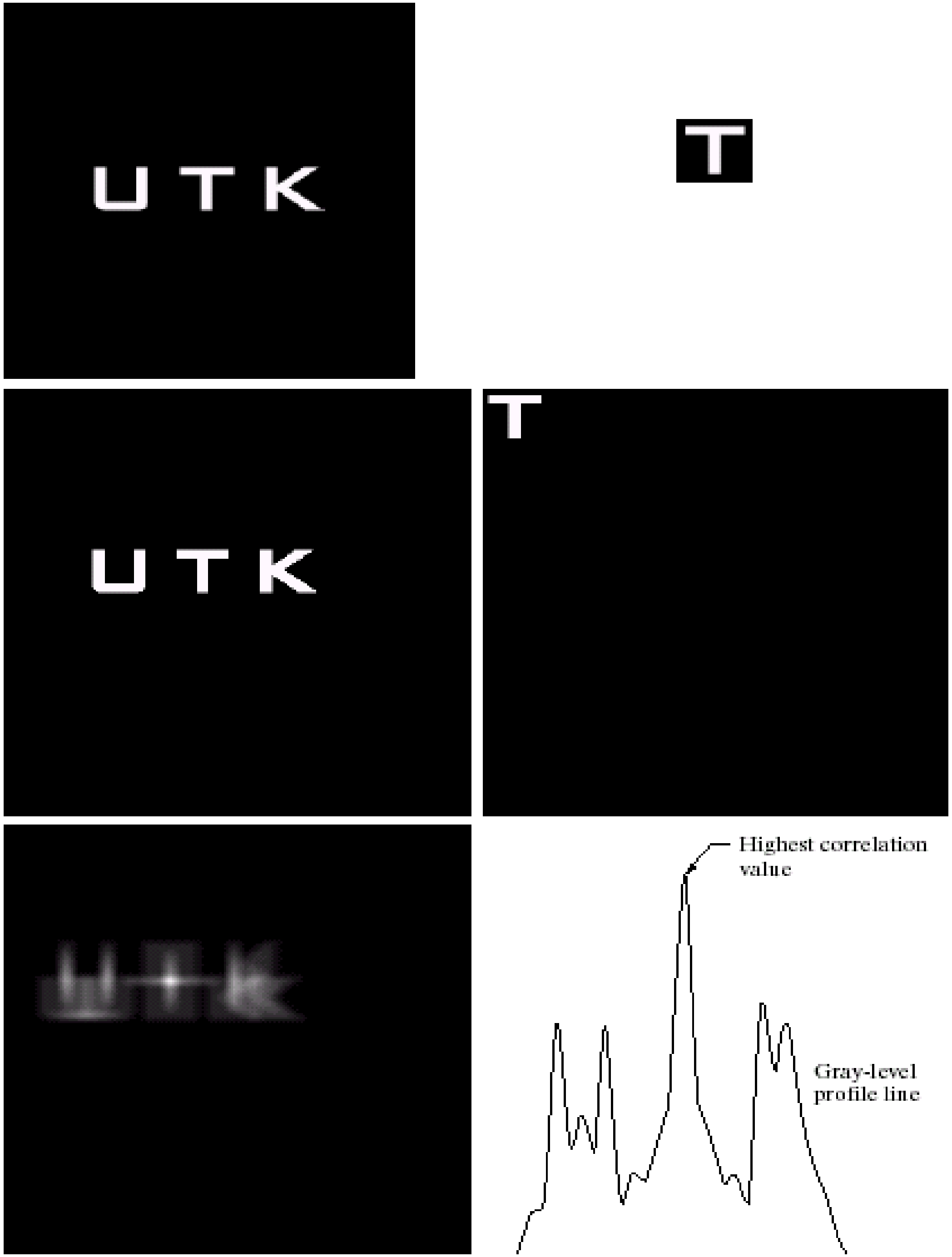
Result of filtering in the frequency domain with properly padded input images.

f(x,y): AxB
h(x,y): CxD
pad zeros to both to make them PxQ

$$P = A + C - 1$$
$$Q = B + D - 1$$

Correlation Example (via DFT)

note the zero padding of both f(x,y) & h(x,y)



a b
c d
e f

FIGURE 4.41 (a) Image. (b) Template. (c) and (d) Padded images. (e) Correlation function displayed as an image. (f) Horizontal profile line through the highest value in (e), showing the point at which the best match took place.

Separability Property of 2-D DFT

- The 2-D DFT can be done by performing row-wise 1-D DFT(or FFT) followed by column-wise 1-D DFT (or FFT), and vice versa.

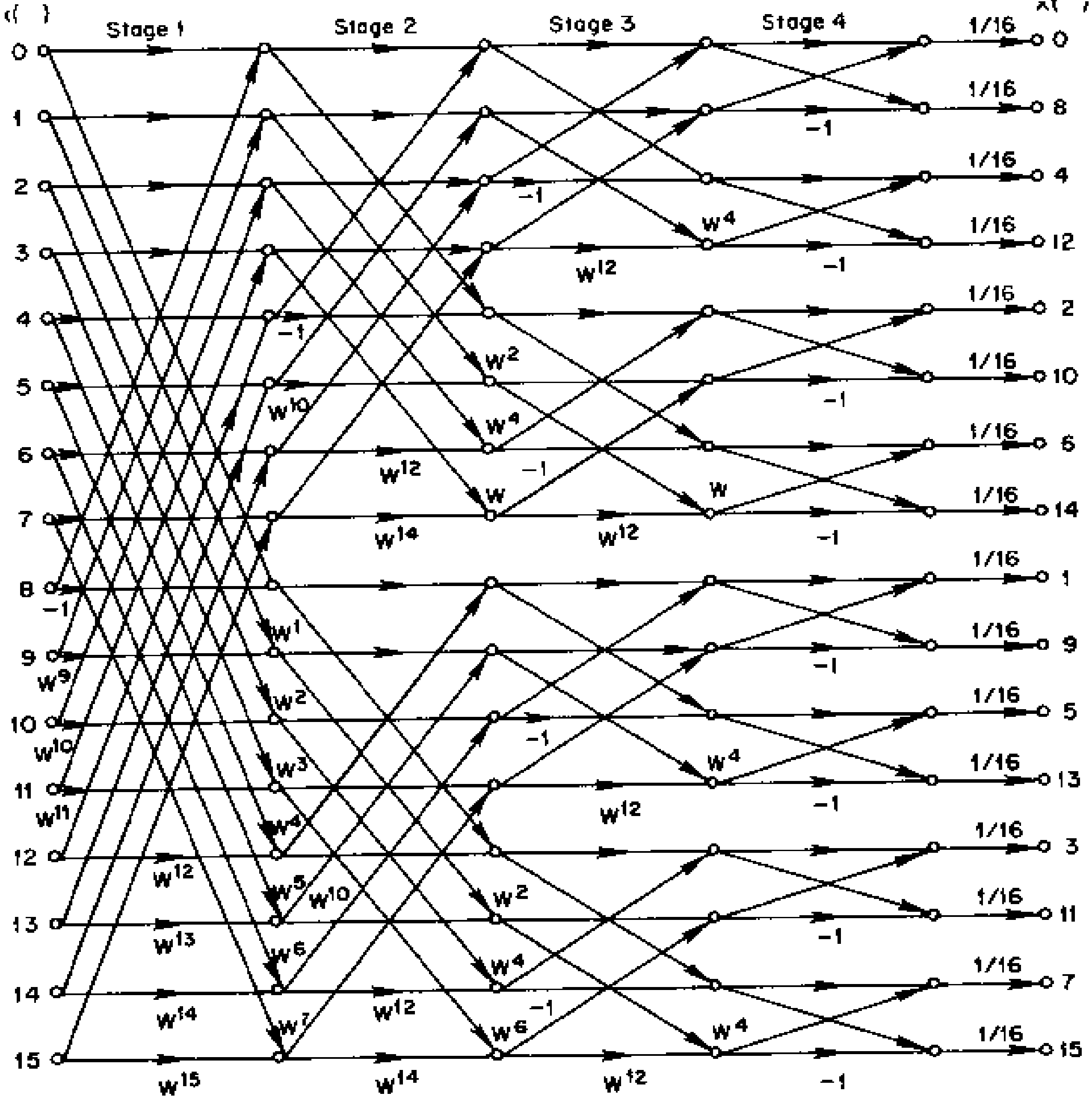
$$F(u,v)=\frac{1}{M}\sum_{x=0}^{M-1} e^{\frac{-j2\pi xu}{M}} \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\, e^{\frac{-j2\pi yv}{N}}$$

$$=\frac{1}{M}\sum_{x=0}^{M-1} F(x,v)\, e^{\frac{-j2\pi xu}{M}},$$

$$where$$

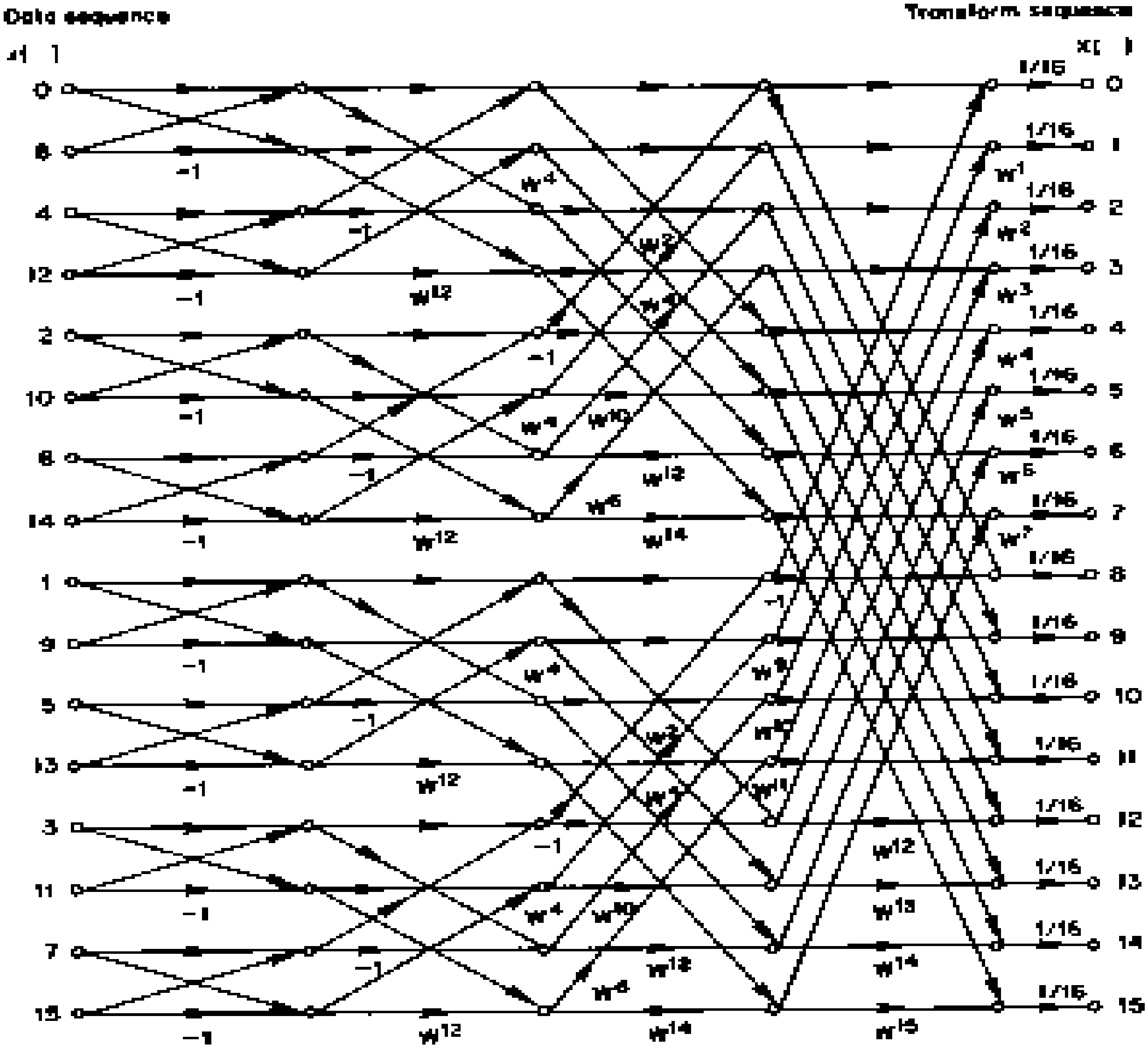
$$F(x,v)=\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\, e^{\frac{-j2\pi yv}{N}}$$

Fast Fourier Transform (FFT)



$$W_N = e^{-j\frac{2\pi}{N}}$$

Fast Fourier Transform (FFT)



Complexity of Fast Fourier Transform (FFT)

- For an N-point transform, DFT requires about N(N-1) complex additions, and N^2 complex multiplications.
  FFT requires about N log N complex additions, and
  (N/2) log N complex multiplications.

- For N=8, DFT requires 64 complex multiplications, while FFT only requires 12 complex multiplications.

Enhancement via Frequency Domain Filtering

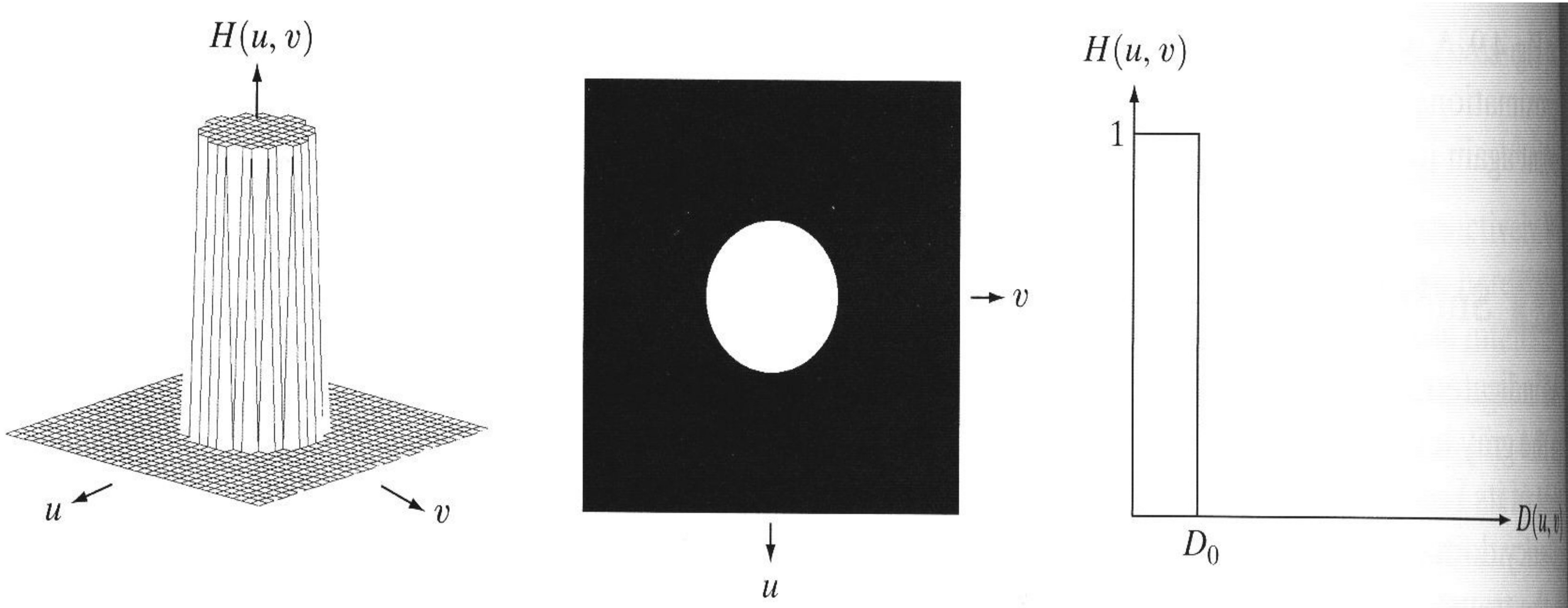
$$y(n_1,n_2)=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}x(k_1,k_2)h(n_1-k_1,n_2-k_2)$$

- The filtering can be done in **discrete frequency domain:**

$$Y(k_1,k_2)=X(k_1,k_2)H(k_1,k_2)$$

- With different **lowpass cutoffs** or highpass cutoffs, different filtering effects can be achieved.
- Ideal LPF in spatial domain is not achievable. Unless we have infinite length $h(n_1,n_2)$, otherwise suffer from Gibbs

Ideal Lowpass Filtering in Frequency Domain



a b c

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

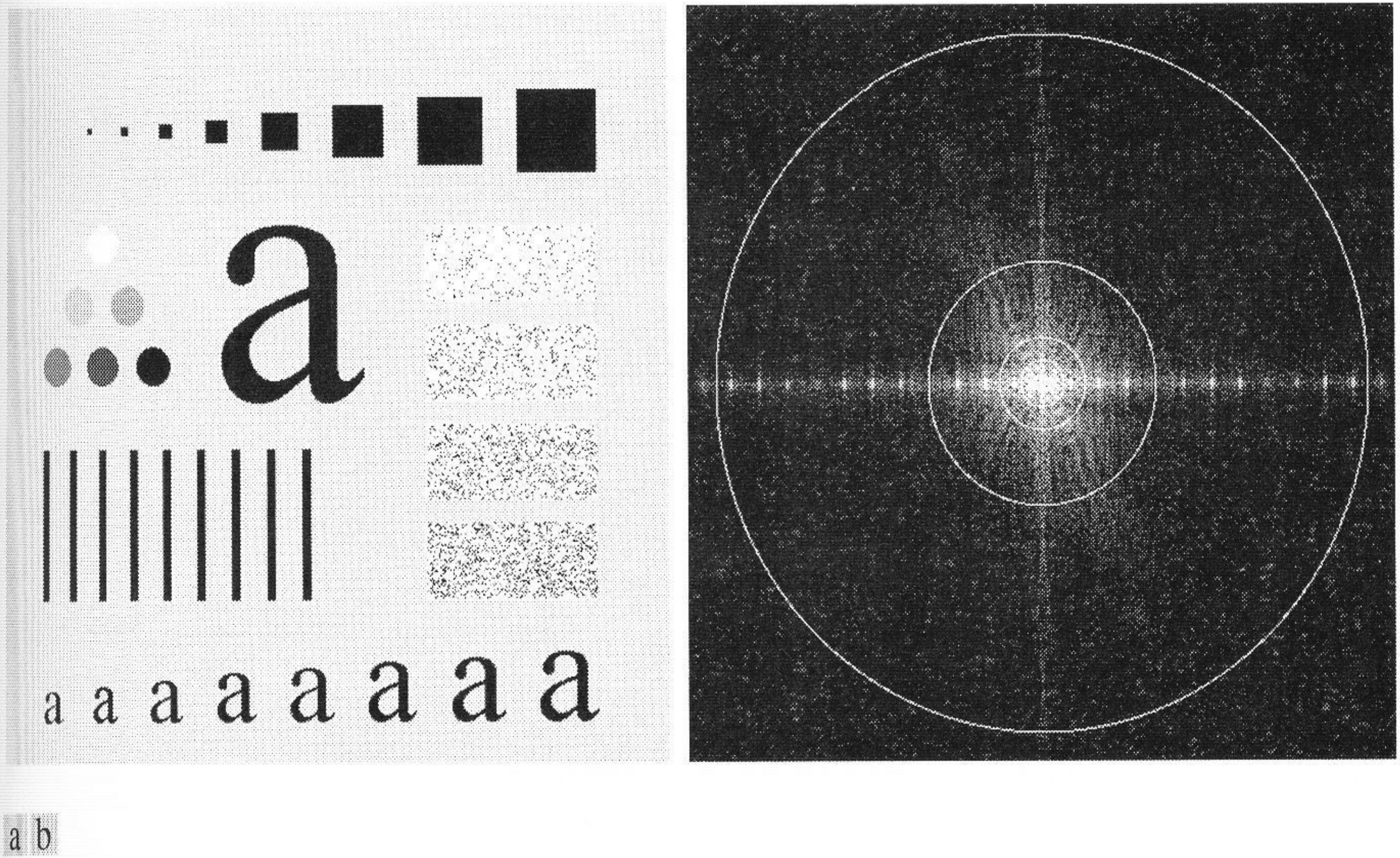
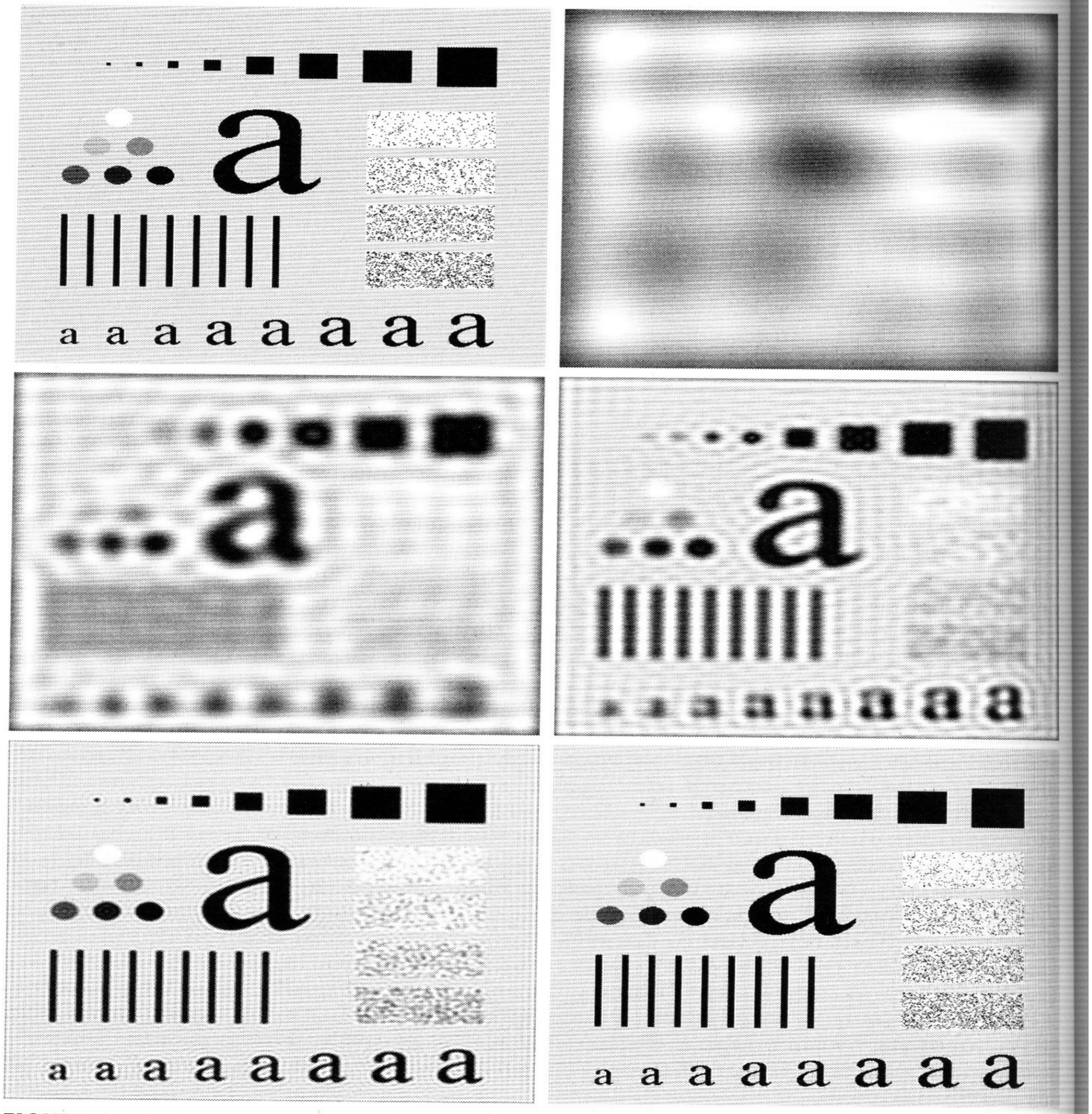


FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Ideal Low-pass Filtering

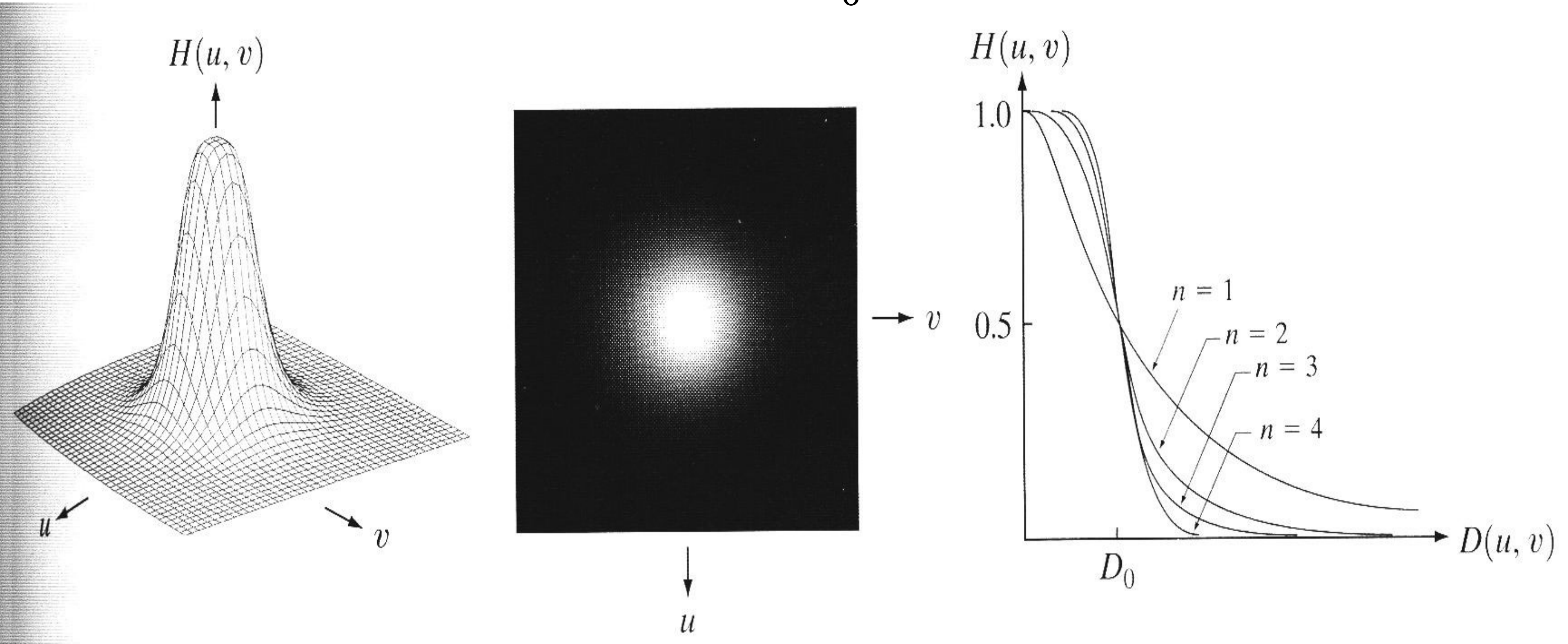


a b
c d
e f

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

Butterworth Lowpass Filters

$$H(u,v)=\frac{1}{1+[D(u,v)/D_0]^{2n}}$$



D(u,v) is the distance from the (0,0) frequency

a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

atial Domain Butterworth Low-Pass Filte

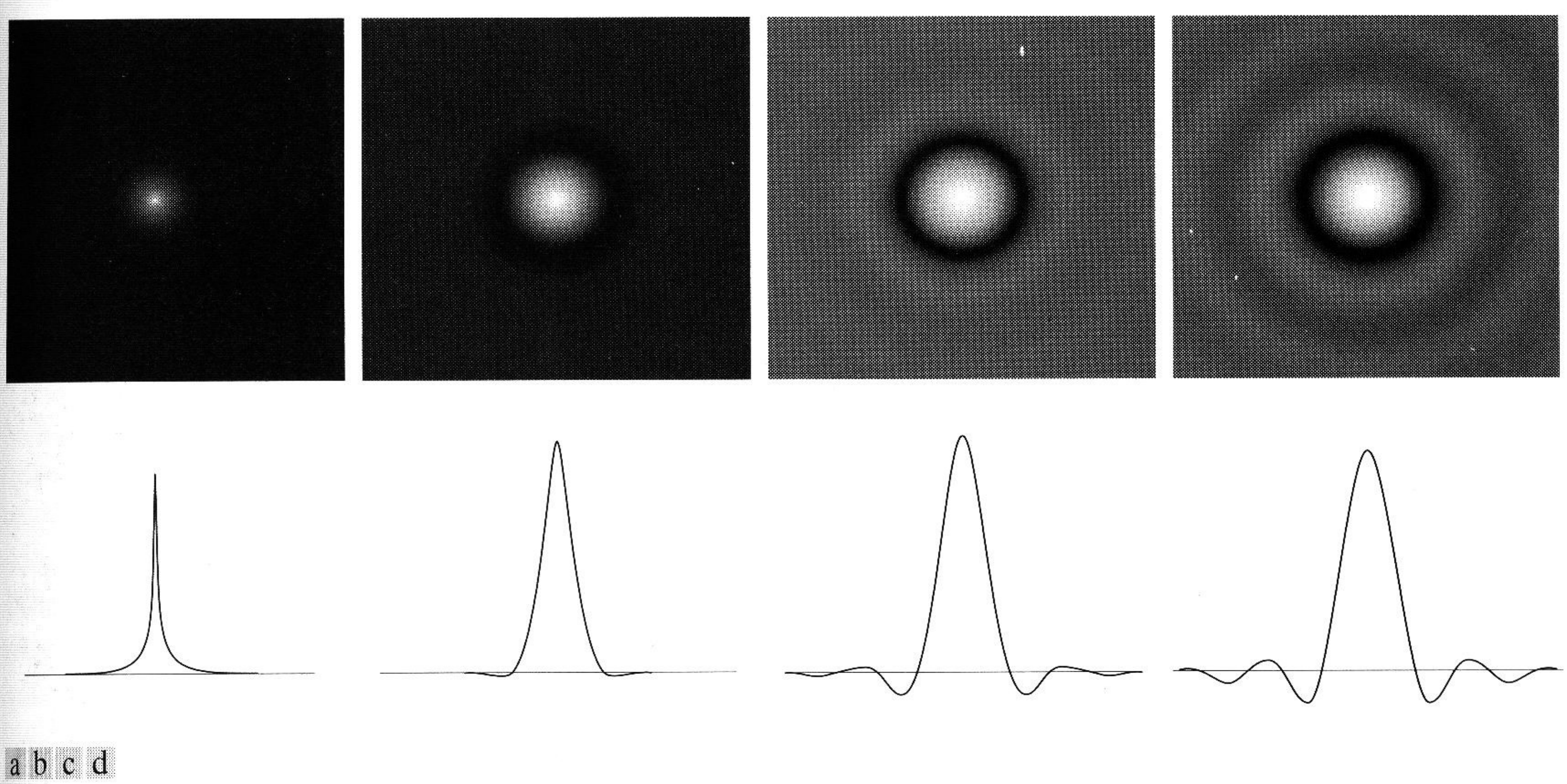
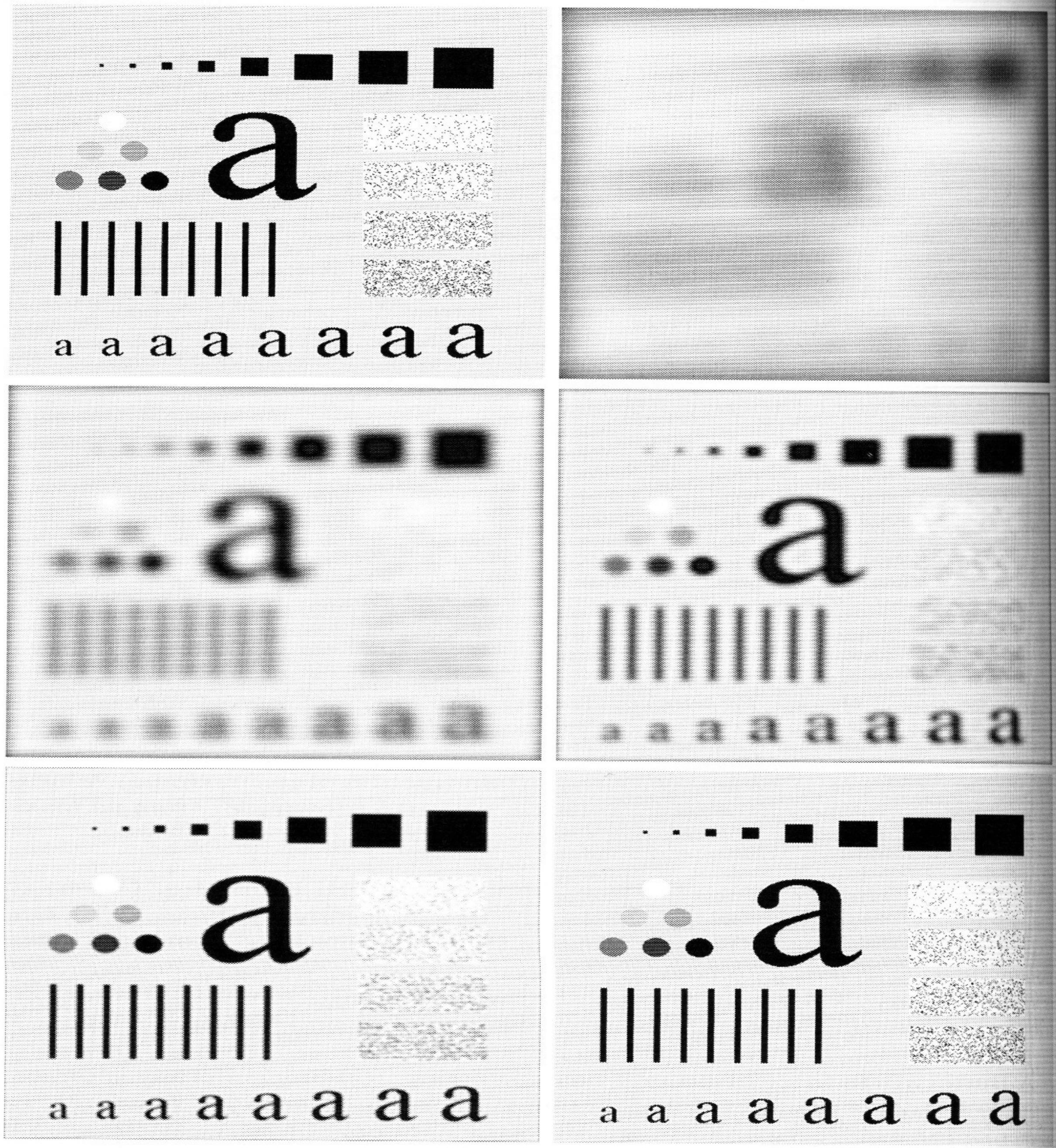


FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Butterworth Low-pass Filtering

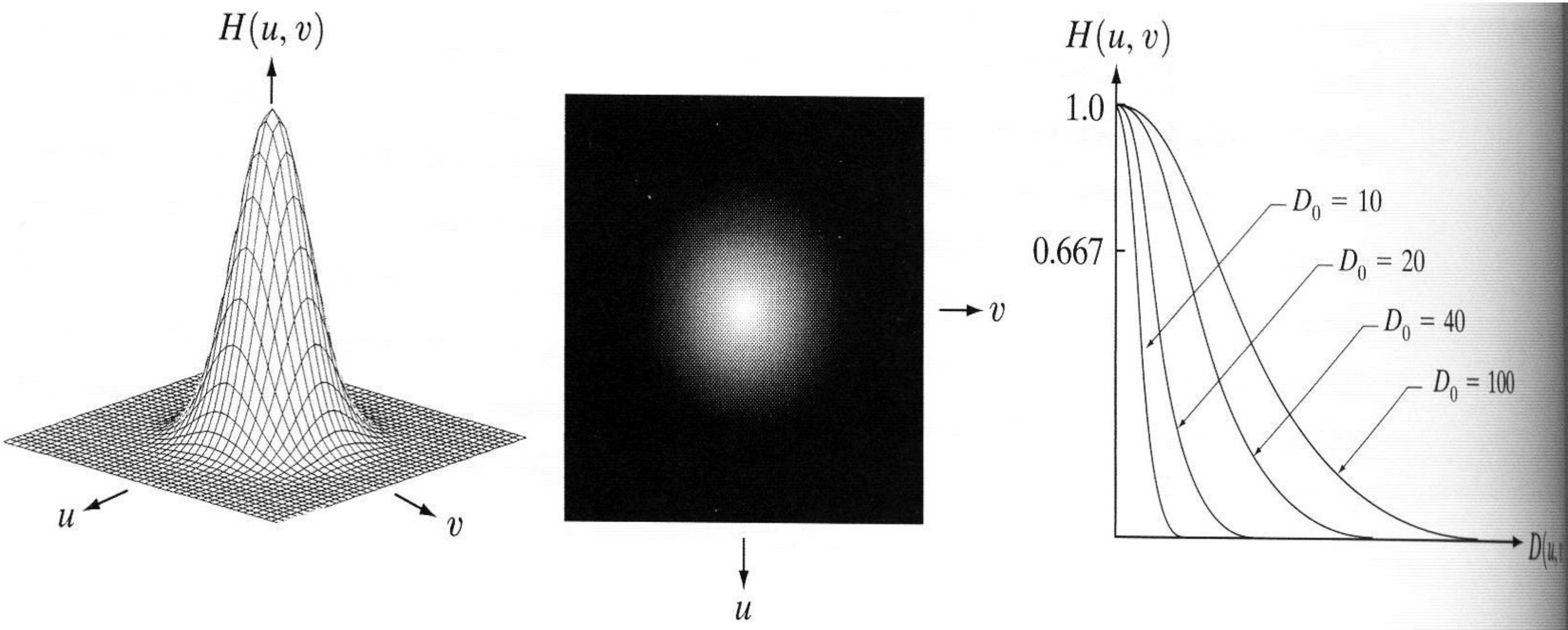


a b
c d
e f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Gaussian Lowpass Filters

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2} = e^{-D^2(u,v)/2D_0^2}$$



a b c

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filt
radial cross sections for various values of D_0.

Gaussian Low-pass Filtering

- The inverse Fourier transform of a Gaussian function is also Gaussian.
- So, a spatial Gaussian filter will have no ringing.

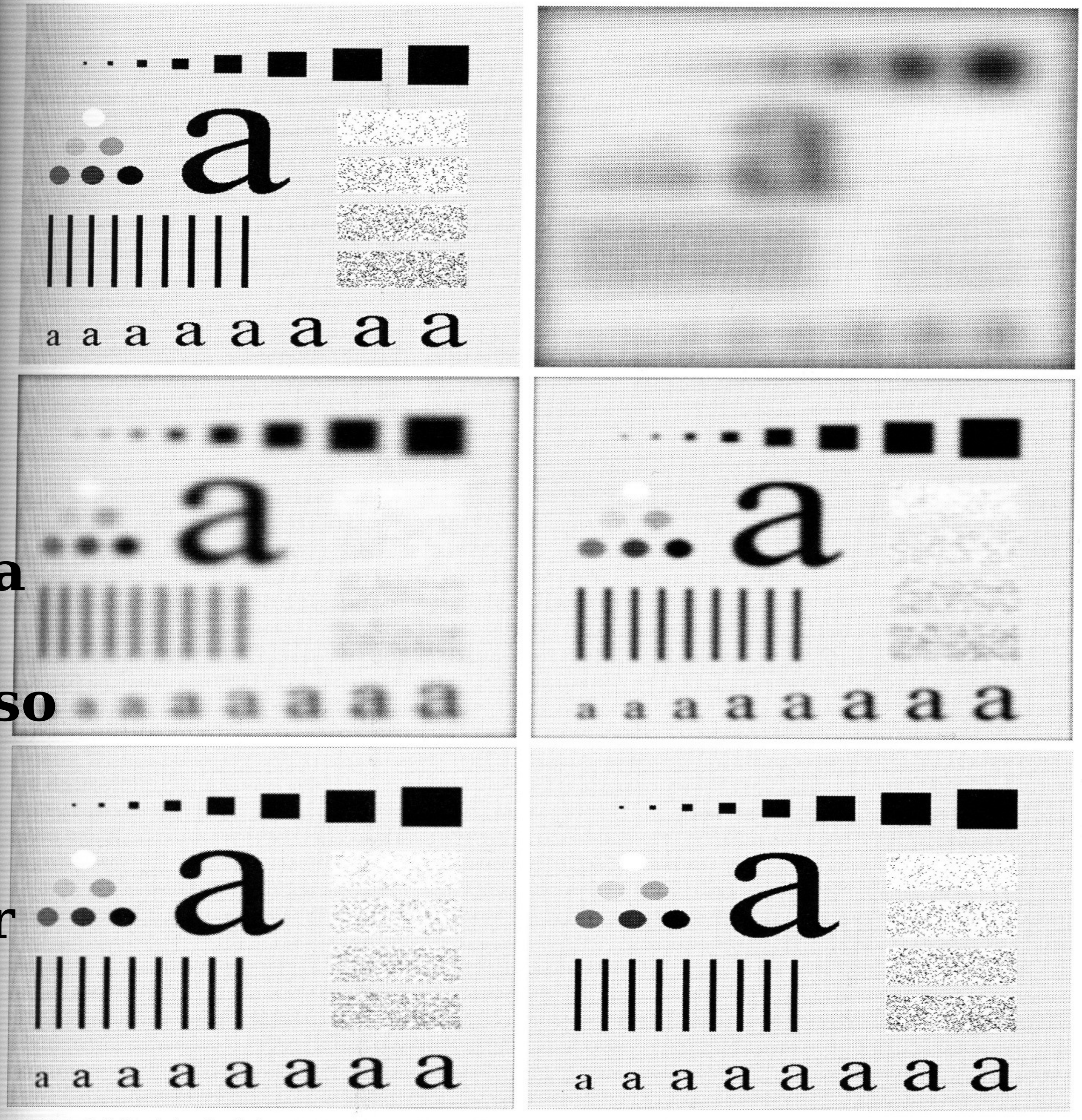
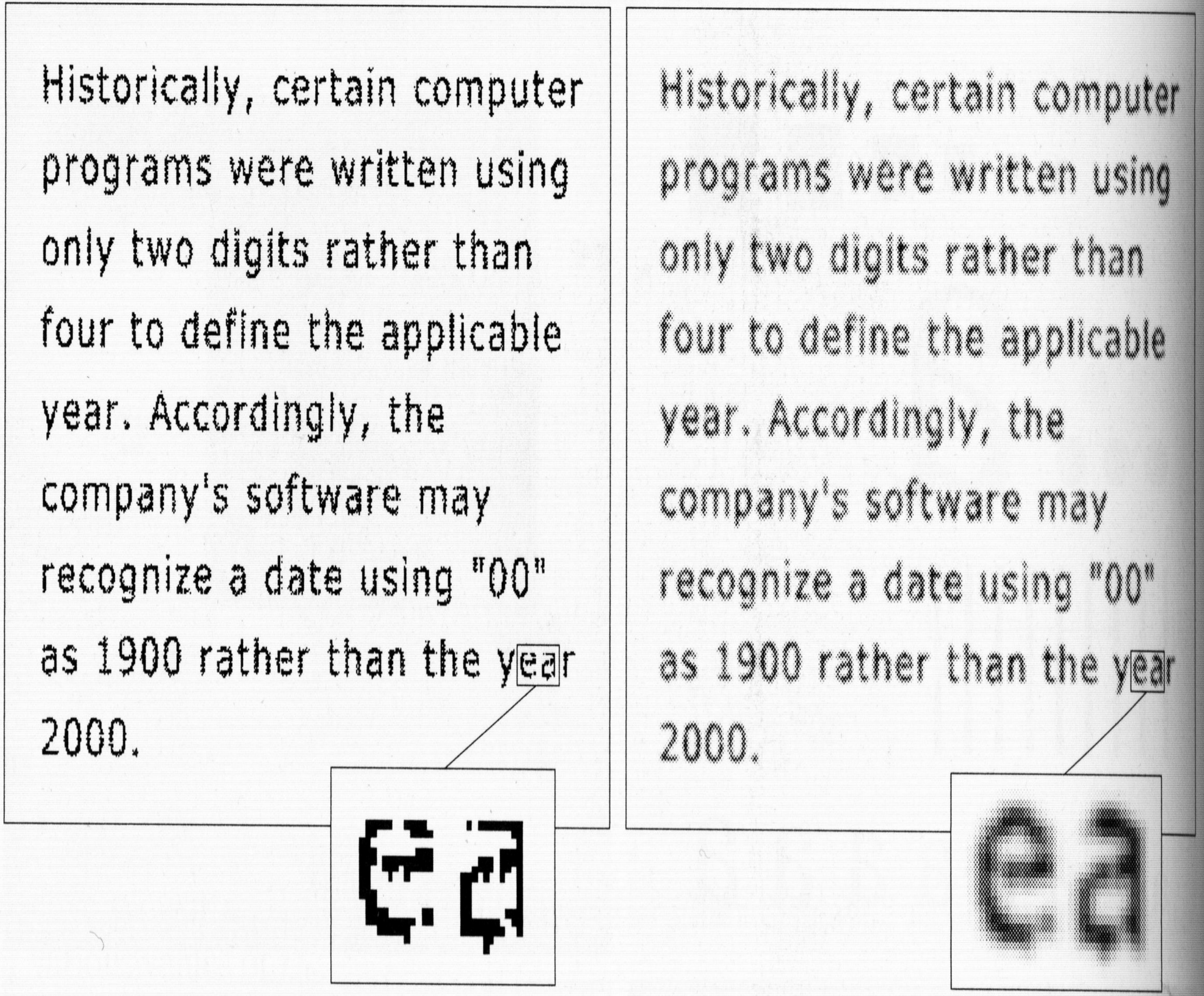


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.



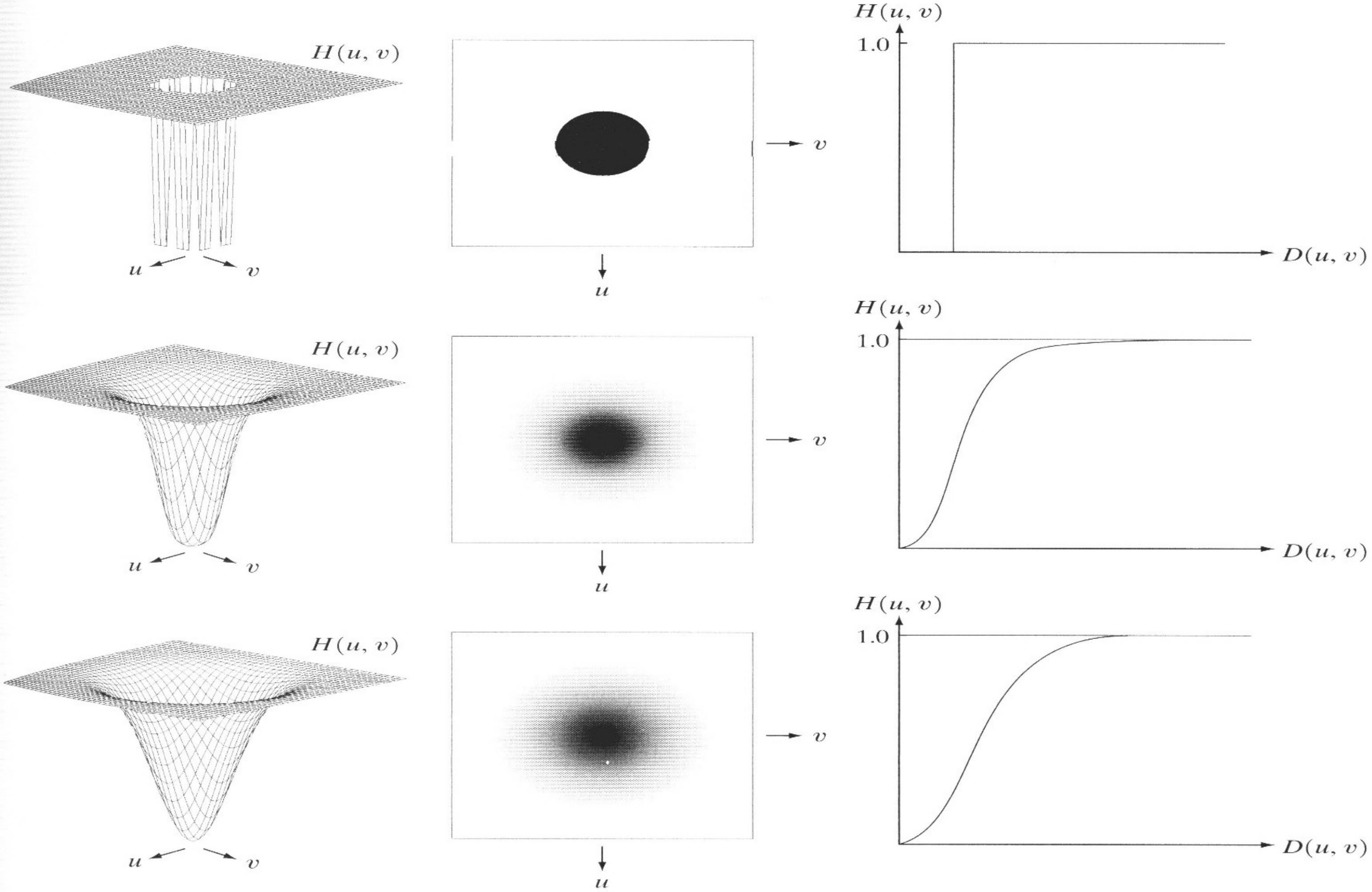
Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.

a b

FIGURE 4.19 (a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

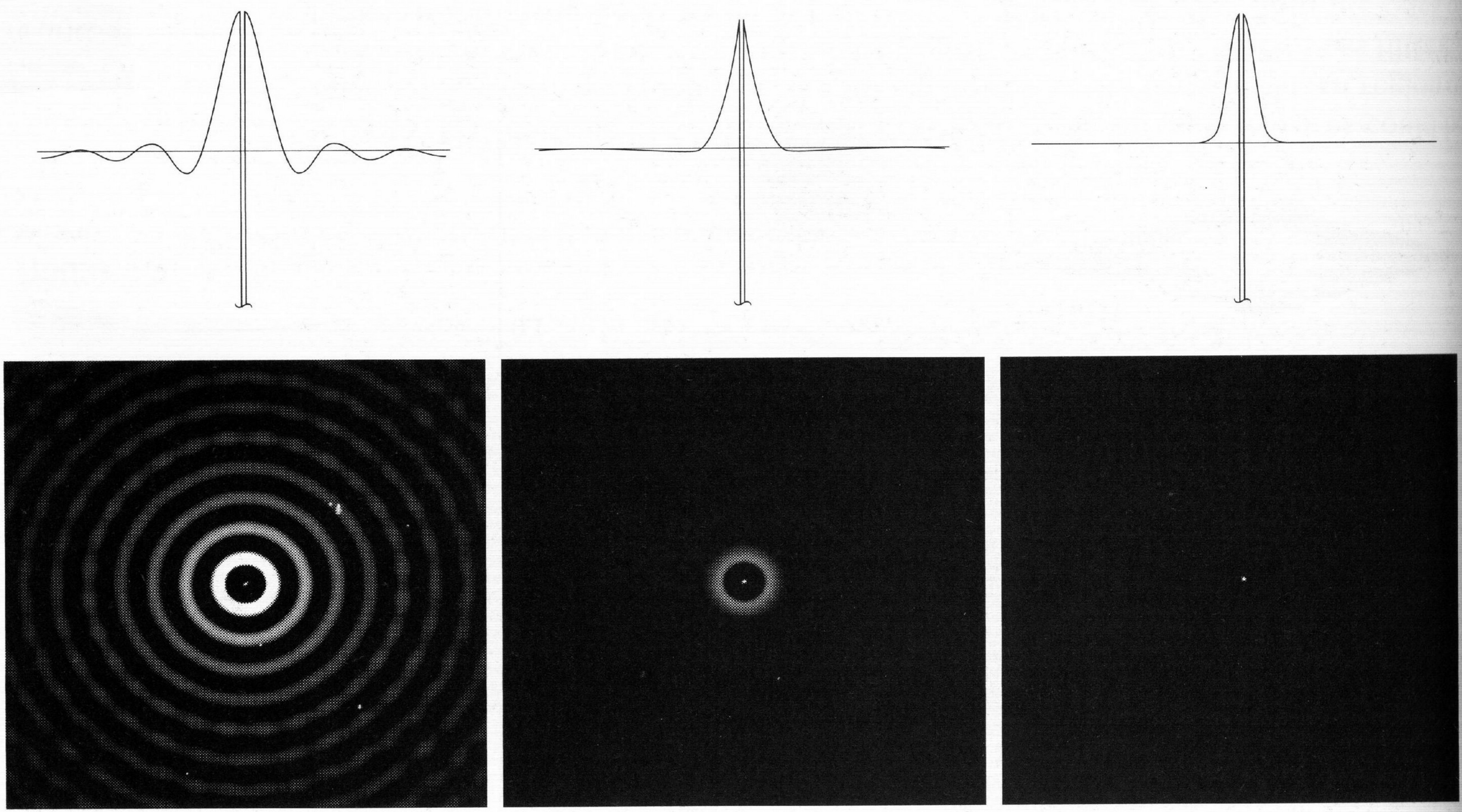
High-Pass Filters $H_{hp}(u,v) = 1 - H_{lp}(u,v)$



a b c
d e f
g h i

FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

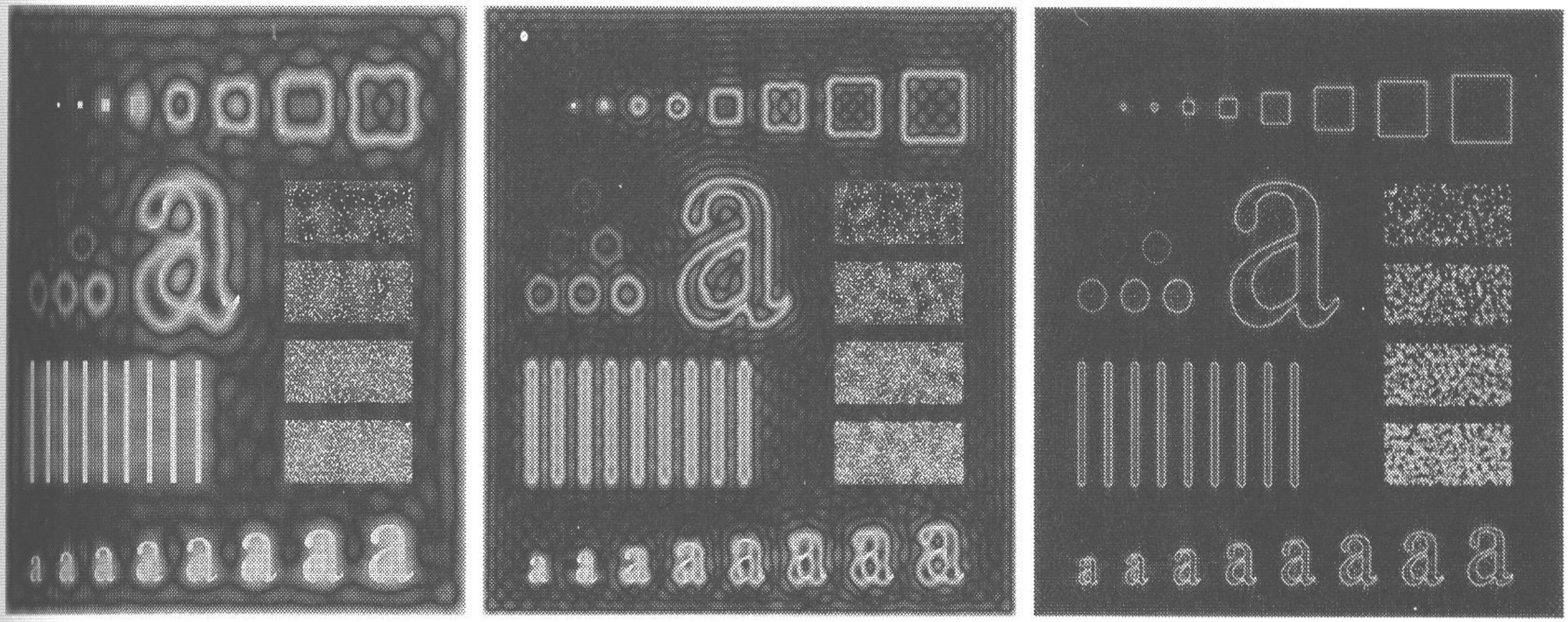
patial Domain High-Pass Filters



a b c

FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

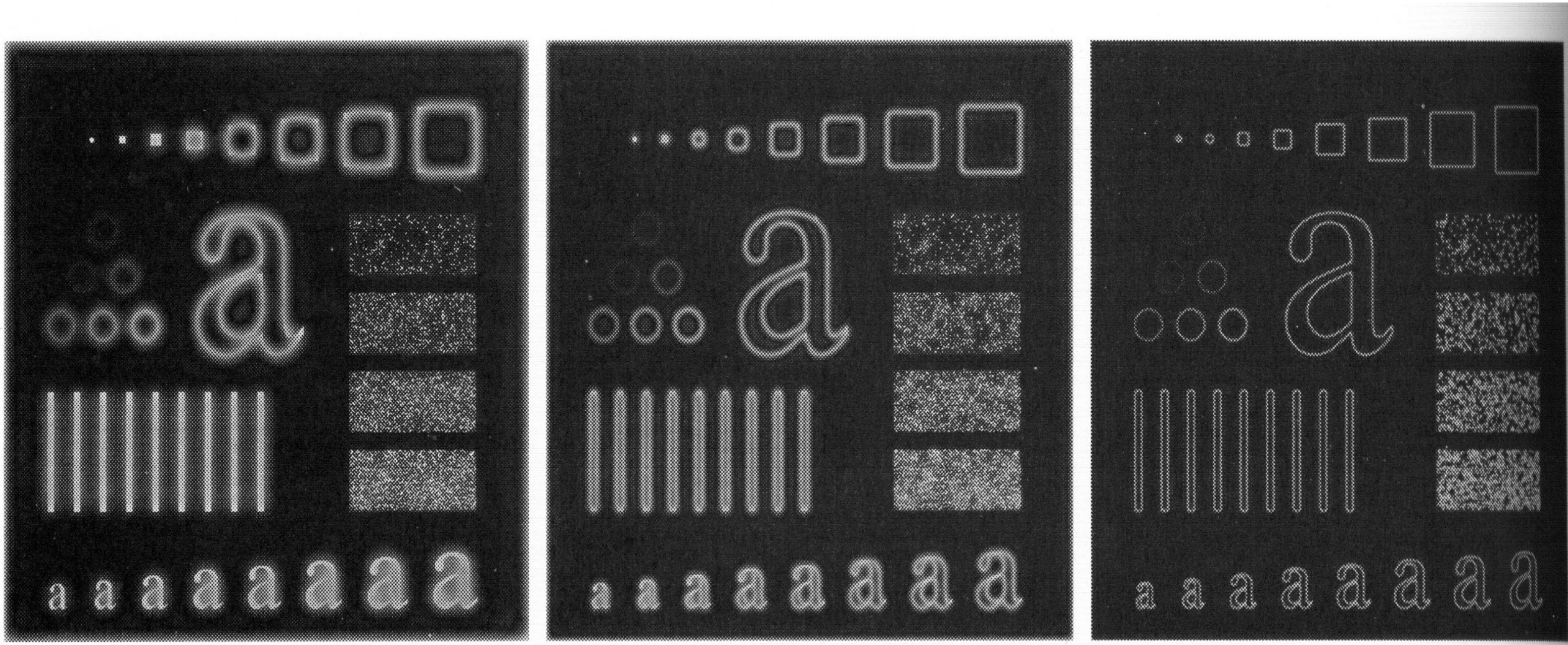
deal High-Pass Filtering



a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

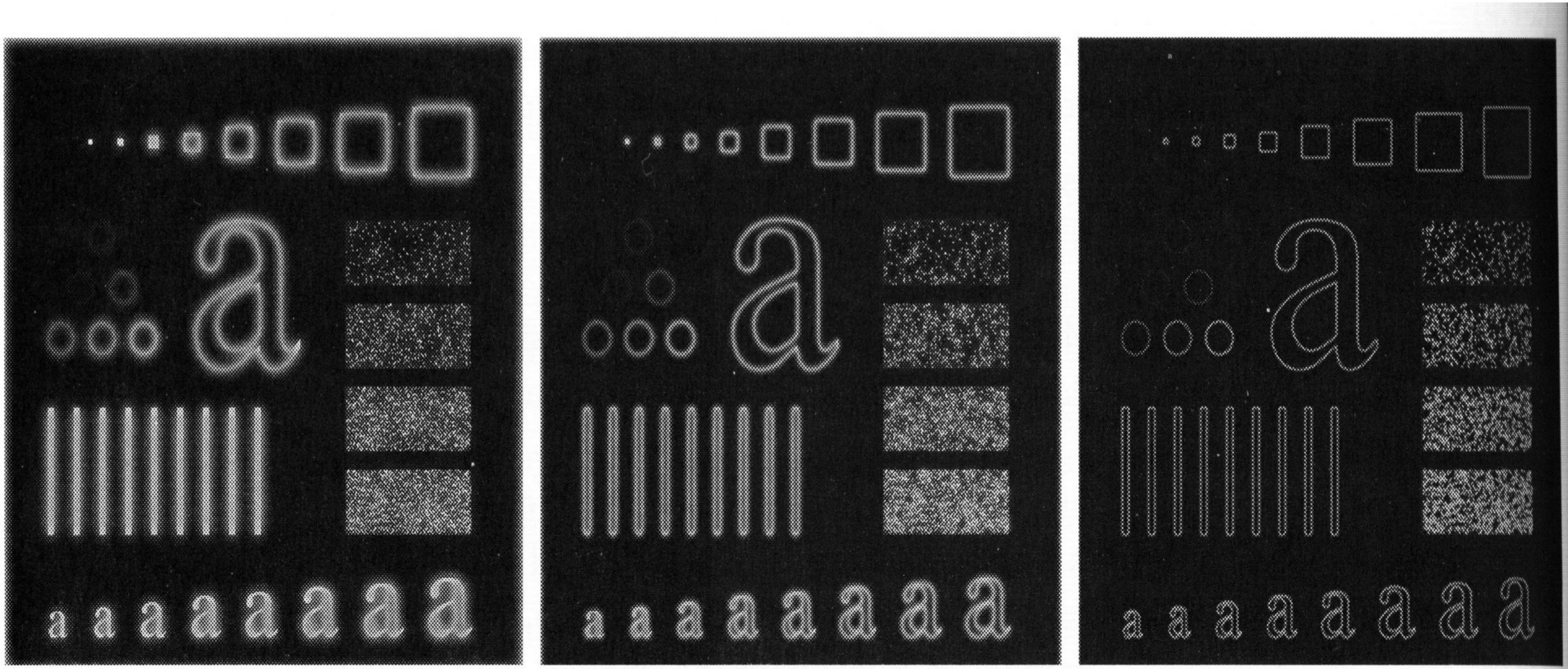
utterworth High-Pass Filtering



a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

aussian High-Pass Filtering



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with D_0 = 15, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

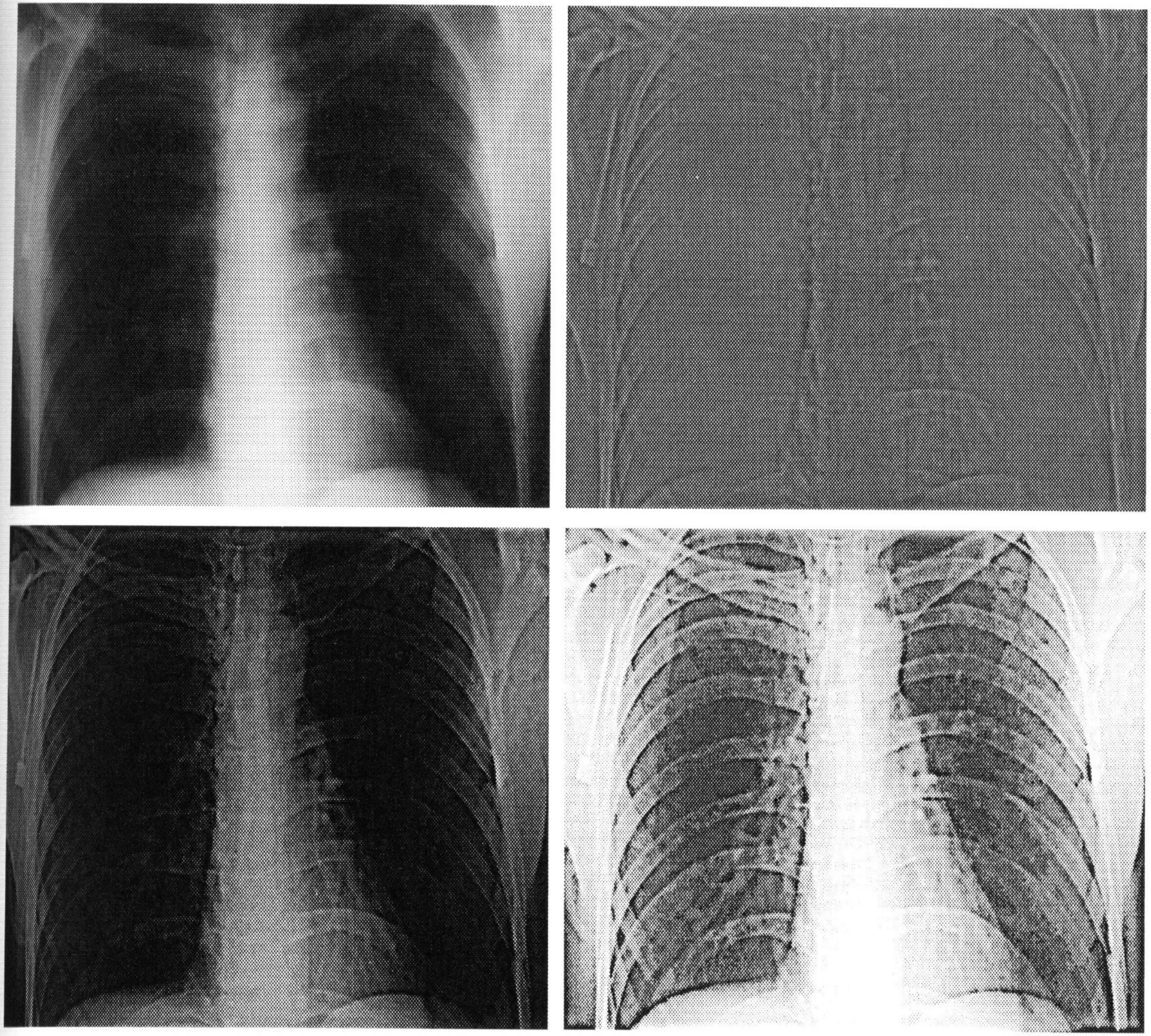


FIGURE 4.30 (a) A chest X-ray image. (b) Result of Butterworth highpass filtering. (c) Result of high-frequency emphasis filtering. (d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

Band-Reject Filters

$$Butterworth\ H(u,v)=\frac{1}{1+[\frac{D(u,v)W}{D^{2}(u,v)-D_{0}^{2}}]^{2n}}$$

$$Gaussian\ H(u,v)=1-e^{-\frac{1}{2}[\frac{D^{2}(u,v)-D_{0}^{2}}{D(u,v)W}]}$$

W is the width of the band

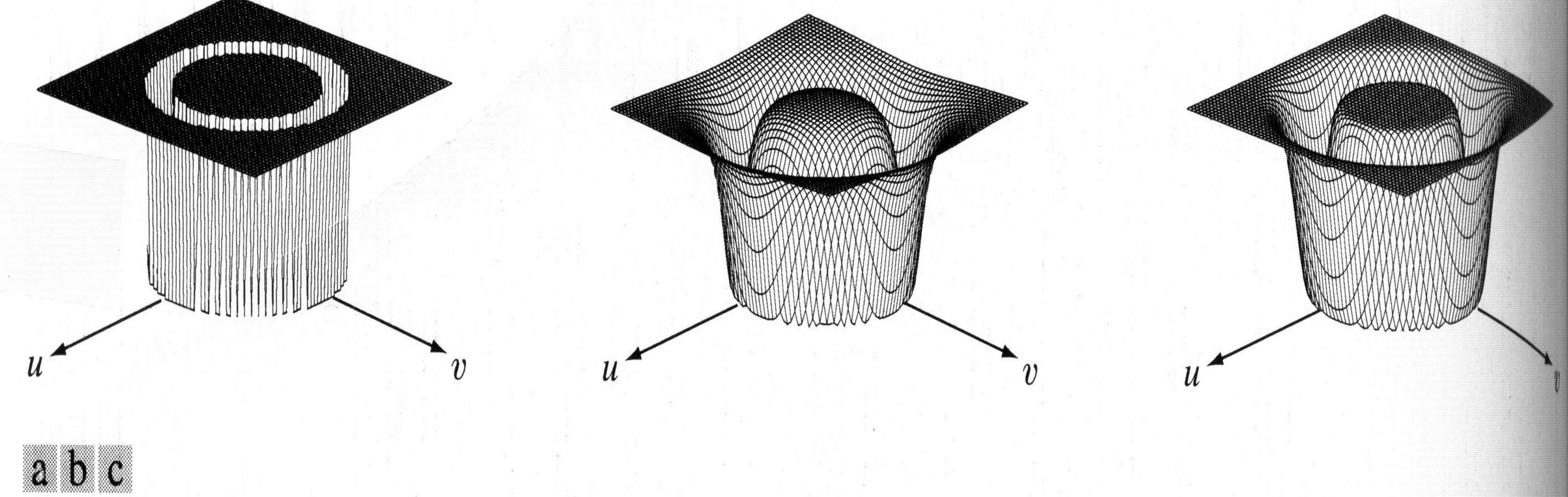
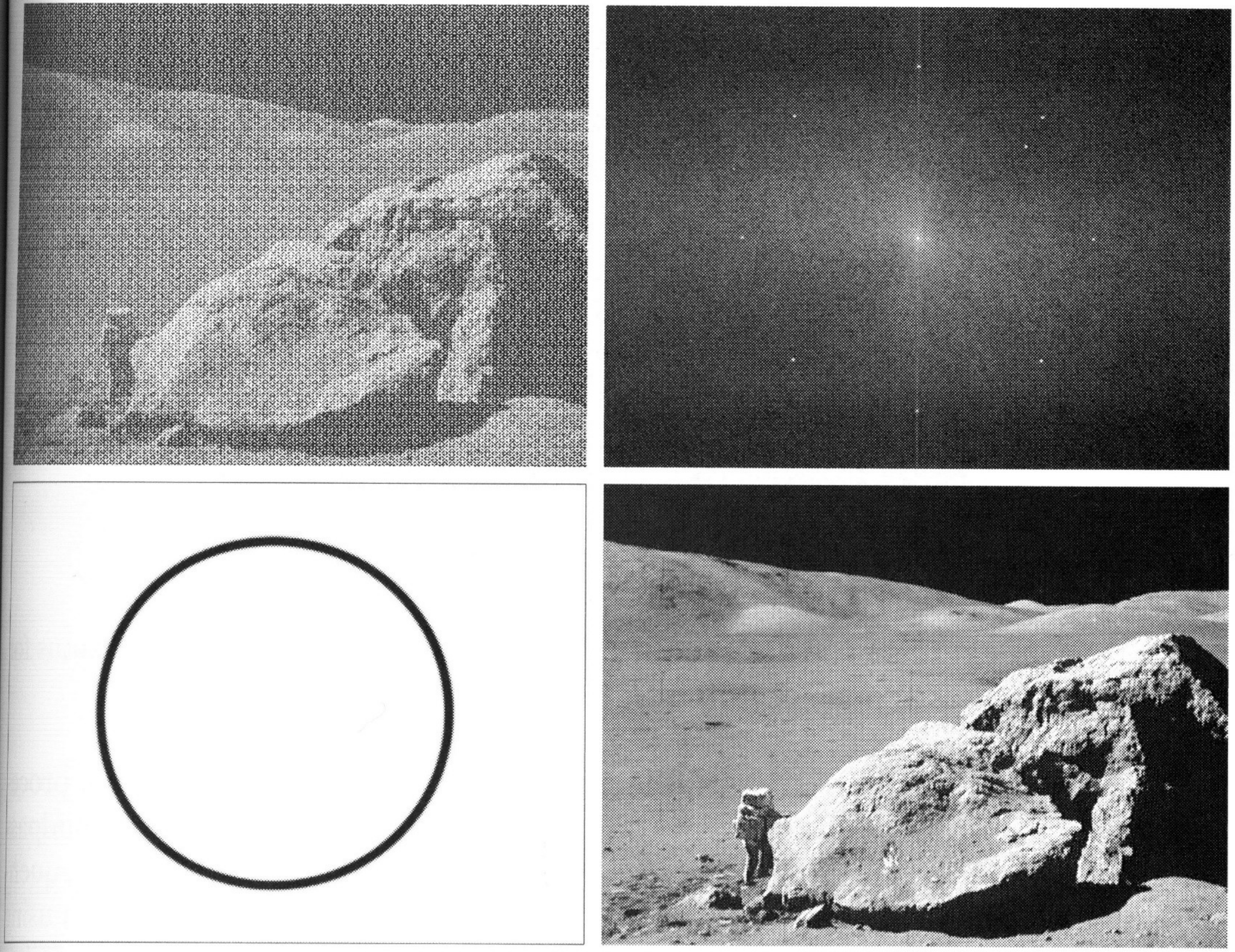


FIGURE 5.15 From left to right, perspective plots of ideal, Butterworth (of order 1), and Gaussian bandreject filters.

Band-Reject Filters



a b
c d

FIGURE 5.16 (a) Image corrupted by sinusoidal noise. (b) Spectrum of (a). (c) Butterworth bandreject filter (white represents 1). (d) Result of filtering. (Original image courtesy of NASA.)

Homomorphic Filtering

- An image $f(n_1,n_2)$ is created by "reflecting" the light from an object that has been “illuminated” by some light source:

$$f(n_1,n_2)=i(n_1,n_2)\bullet r(n_1,n_2)$$

- The illuminating intensity $i(n_1,n_2)$ is a slowly varying component (controlling the overall dynamic range),

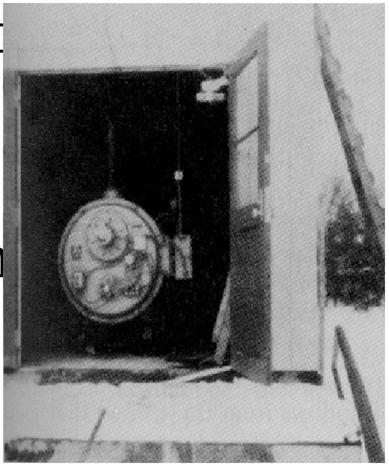
  $r(n_1,n_2)$ is a fast varying component (affecting the local contrast).

- We like to decrease the low-frequenc

  and amplify the high frequencies

  (simultaneous dynamic range

  compression and contrast enhancem



Homomorphic Filtering

- **Separating** these two components and de-emphasize/emphasize each component:

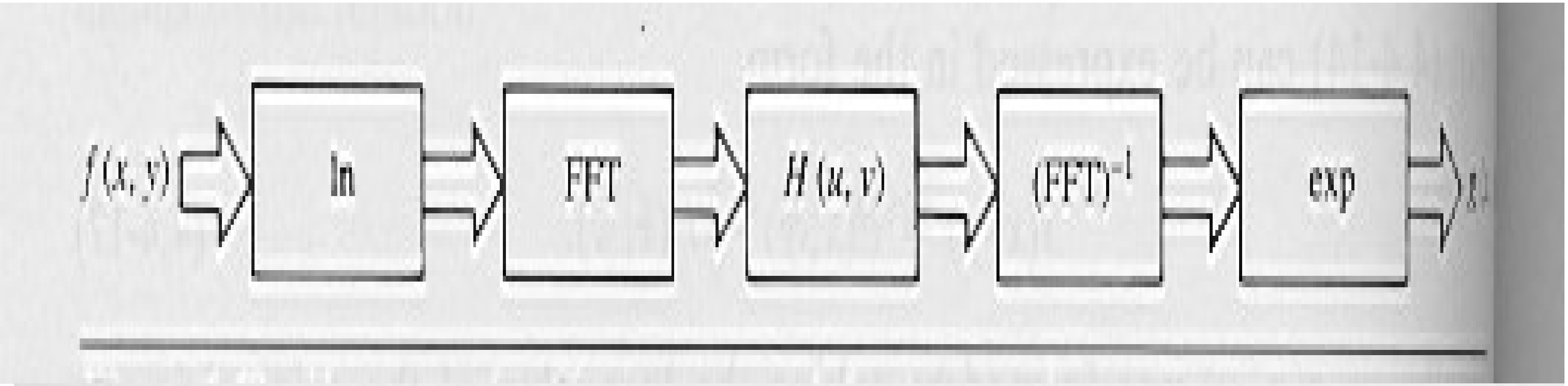
$$\ln x(n_1, n_2) = \ln i(n_1, n_2) + \ln r(n_1, n_2)$$

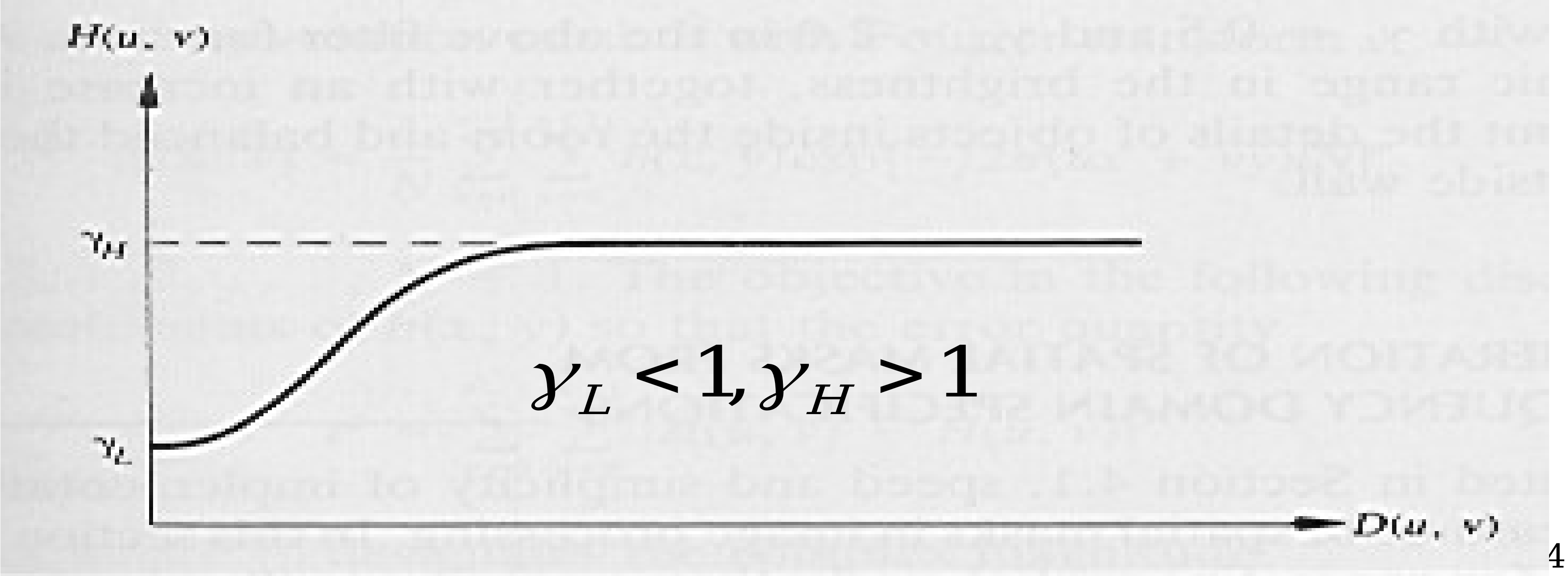
- Take Fourier Transform

$$Z(u,v) = I(u,v) + R(u,v)$$

- Filter with a filter function H(u,v) that de-emphasize the low-frequency and emphasize the high-frequency component.

Homomorphic Filtering

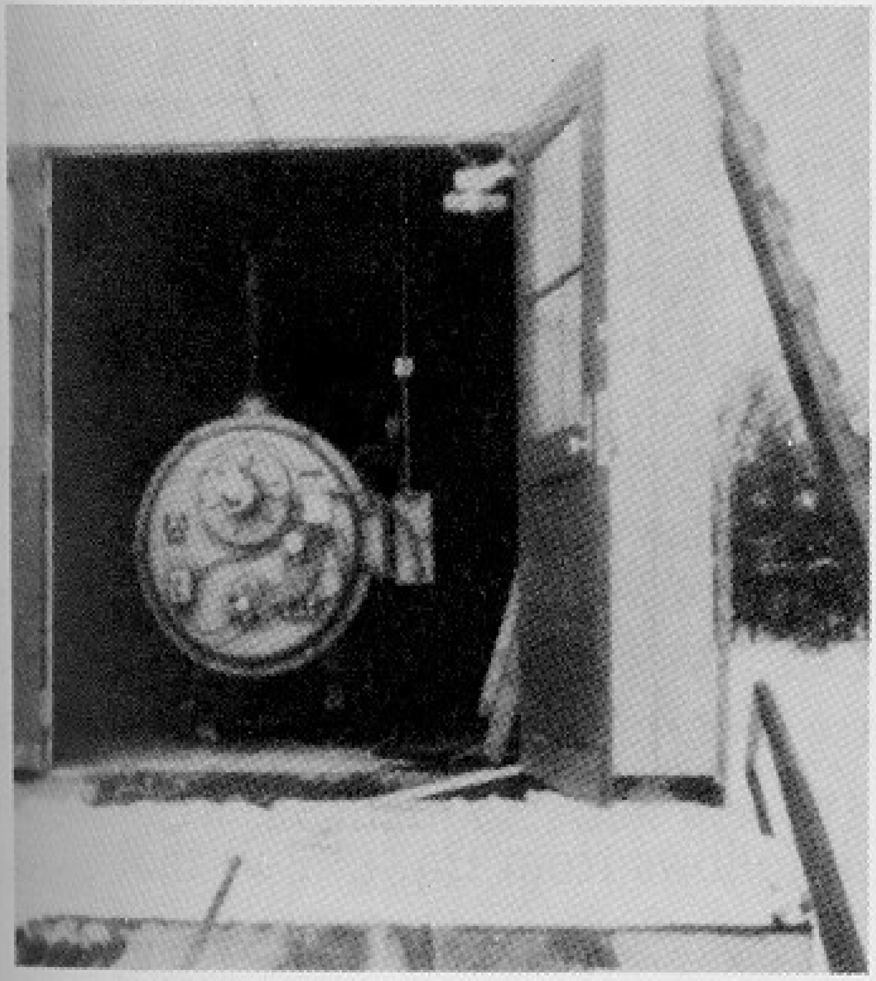




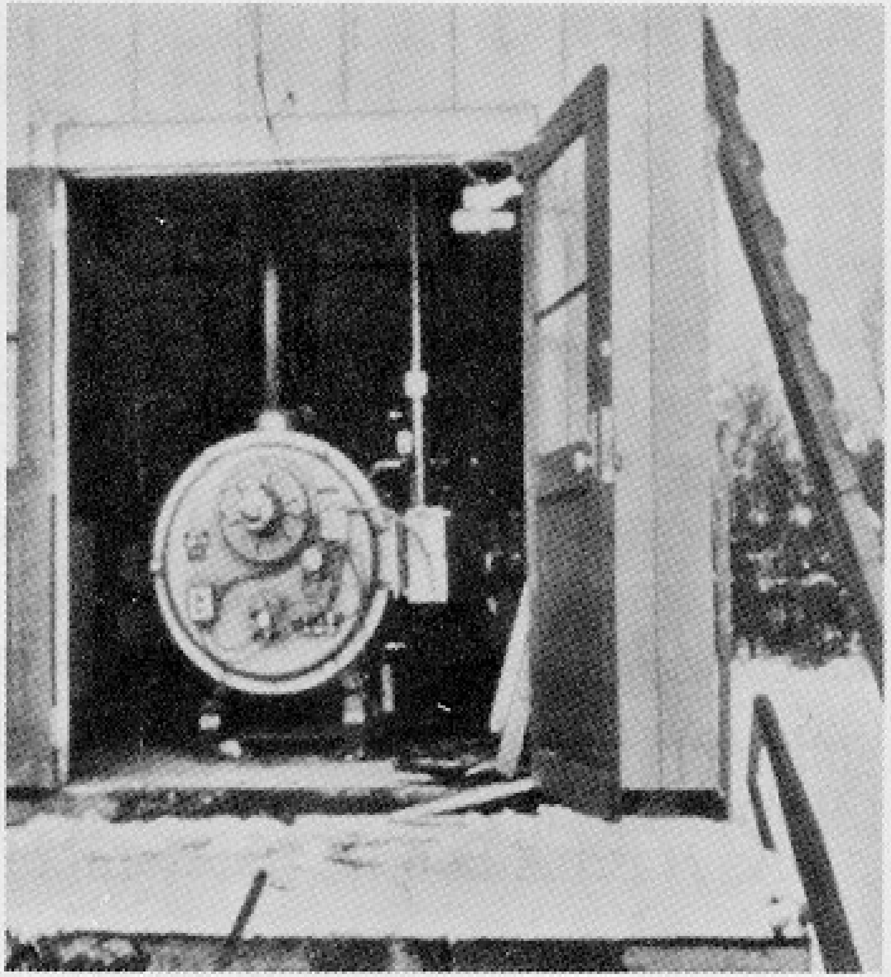
$$\gamma_L < 1, \gamma_H > 1$$

Original & Homomorphic Processed Results

$$\gamma_L = 0.5, \gamma_H = 2.0$$



(a)



(b)